On the Incentives to Exacerbate Polarization*

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Abstract

We consider an auction involving bidders who are "polarized." There are three potential bidders, a moderate or neutral bidder, and two bidders who are polarized in the sense that they prefer the neutral bidder to win the auction rather than the other polarized bidder. The seller cannot commit to an optimal mechanism, but can decide which bidders to allow to participate. While greater competition is generally thought to be beneficial for the seller, we identify conditions under which the seller can increase her expected revenue by preventing the neutral bidder from participating. By excluding the neutral bidder, the seller increases the willingness to pay of the polarized bidders. Thus, rather than seeking to bring about compromise, our analysis suggests organizers have an incentive to exacerbate conflict. While potentially revenue enhancing, excluding the neutral bidder always makes the auction less efficient; in fact, the incentive to exclude her is greatest precisely when it is most harmful from a welfare perspective. We discuss applications of our model in economics and politics.

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1. Introduction

In many situations an organizer will bring together multiple competing interests that seek to influence the final outcome. Countries negotiate agreements over trade, the practice of war, and the environment; political leaders bargain over budgets; elected officials consult and bargain with interest groups; companies buy and sell assets; firms in an industry determine standards among themselves and with the government. A key consideration for the organizer, when a decision will affect many individuals or groups, is whom to invite to the table to participate in the discussions and offer resources to affect the outcome. This is the question we consider in this paper.

We model the organizer as a revenue maximizing seller who runs an auction that allocates a contract (for example, control over a policy or an asset) to the highest bidder, but who can restrict the set of eligible or qualified bidders. Bidders have private direct valuations for the contract, but also differ in the externalities they impose on other bidders. When the identity of the winner affects the level of negative externality experienced by other bidders, we say that the environment is *polarized*. We assume there is a neutral bidder, who imposes minimal externalities, and two polarized bidders who impose larger externalities and are harmed when the other polarized bidder wins. Due to these identity-dependent externalities, a bidder's willingness to pay depends not only on her own valuation but also on who she would be likely to lose to if she does not secure the contract for herself.

We are interested in environments in which the seller lacks the commitment power to run an optimal auction, and so consider standard auctions in which the highest bidder wins, with a particular focus on the second-price auction. Before the auction begins however the seller can publicly commit to the set of bidders who are eligible to participate. We study an organizer's incentive to exploit the threat of losing to a particular bidder by strategically excluding compromise alternatives. As the term polarized implies, ideological politics are a particularly salient example. The threat of having policy set by an ideologically more distant competitor is greater than the threat of having policy chosen by a moderate. However, as examples from industry, government, and sports discussed below will illustrate, polarized competition between participants can emerge in other settings when outcomes generate competitive spillovers.

When deciding whom to invite, the organizer must consider two distinct effects that changes in the pool of bidders have on revenue. The *competition effect* depends on the number of bidders invited and is unambiguous and straightforward: as the number of bidders increase, the organizer is able to collect more bids and hence more draws from the distribution of private values. Consequently, the competition effect of increasing the number of bidders always increases the revenue of the organizer.

The second effect, which we refer to as the *stakes effect*, depends on the makeup of the bidder pool and has ambiguous effects on revenue that depend on who is invited. Excluding a bidder alters the calculus for the invited bidders by altering the expected cost of losing. We say that stakes are raised (*resp.* lowered) for a bidder if the expected cost of losing are increased (*resp.* reduced) by excluding a bidder. Excluding a polarized bidder always weakly lowers the stakes for the remaining bidder and hence decreases their bids. Thus, the stakes effect of excluding a polarized bidder is negative. As the competition effect of excluding a bidder is always negative, excluding a polarized bidder is never beneficial for the organizer. However, when the neutral bidder is excluded, the stakes are raised for the polarized bidders: when it is no longer possible to lose to the neutral bidder, the disutility from failing to win the contract is higher. Consequently, the stakes effect of excluding the neutral bidder is revenue positive.

Whether or not the organizer benefits from including the neutral bidder depends on whether the competition or stakes effect is stronger. When private values are more widely dispersed, the increase in the number of private values draws afforded by including the neutral bidder is more valuable and the competition effect is stronger. On the other hand, the stakes effect is stronger when polarized bidders are more strongly opposed to losing to each other. Our model thus predicts that when valuations are widely dispersed or polarization is low, the neutral bidder is likely to be included. Conversely, in environments with low variation in valuations, or high polarization, the organizer will find it optimal to exclude the neutral bidder. However, it is precisely in environments with high polarization that allocating the contract to the neutral bidder is most likely to be welfare maximizing. Our analysis thus points to a fundamental conflict between the organizer's interests and society's welfare. Rather than internalizing the negative externalities that polarized bidders impose on each other, the organizer seeks to exploit these costs by playing bidders off against each other. So, rather than finding common ground or bringing about compromise, our analysis suggests there is a strategic incentive for organizers to exacerbate conflict.

Restrictions on participation are characteristic of many negotiations: international trade agreements are often regional in scope; special interests are invited to or excluded from political negotiations; professional athletes limit the pool of teams they will consider; companies often create competitions for investment among a limited set of sites; asset sales often limit the pool of qualified participants. Many of these environments have an element of polarization between the interests. Ideological differences are one source of polarization, but more generally polarization can arise whenever prevailing in the auction can alter the competitive balance between the bidders in the future.

Of course, for the seller to ever benefit from restricting the set of bidders who can participate, it must be that the seller cannot design the optimal auction, but rather is constrained to a standard auction in which the highest bidder is awarded the contract. If the seller could commit to the optimal auction, she could always do weakly better by having any additional bidder included. This lack of commitment is reasonable in many circumstances, including in environments in which the seller can select the pool of bidders. For example, in government procurement auctions the agency first decides which firms are eligible to bid, then is legally obligated to buy from the firm offering the best price (Kang and Miller, 2017). Similarly, a standard way to model lobbying (Baye, Kovenock, and De Vries, 1993; Fang, 2002) is that the government first chooses which lobbyists are permitted to participate, then runs a standard auction among this set of bidders.

The key elements of our model are that the seller can choose the set of bidders eligible to participate, that this set is publicly observed, and that there is no opportunity for re-sale. If we interpret the bidding as lobbying in order to affect the government's policy then resale would not be possible. For this application the neutral bidder would correspond to a moderate interest group, and the strategic incentive to exclude them would, in turn, reduce the incentive of moderate interest groups to organize in the first place. While others have suggested that a lack of resources or a collective action problem prevents moderate interests from organizing (Olson, 1965), the incentive to strategically exclude moderates could also contribute—after all, there is little reason to organize if you still would not be given the opportunity to advocate for your objectives. Our analysis then implies that actions aimed at solving the collective action problem may not be enough to get moderates to the table.

Beyond lobbying, there are many settings in which different candidates make proposals and in which the set of contenders is publicly narrowed by the organizer over time. One such environment involves cities bidding to be chosen as the site for a major sporting event (such as the Olympics or World Cup) or for a corporate office (such as Amazon's second headquarters).[1] Cities are likely to care who they lose to—due to spillovers between locations or, in the case of the Olympics, because the IOC tends to alternate continents—and we demonstrate that

[1] While Amazon's second headquarters was a particularly prominent example, city and state governments competing for corporate investments is a common process. See Kim (2018) for an empirical analysis of this bidding that does not consider externalities.

narrowing the alternatives can increase what the organizer can extract from the proposals.[2]

Another market in which the value of obtaining a contract includes both an intrinsic value for the asset and strategic concerns for preventing a rival from winning is the market for advertising. The value of placing an ad reflects not only the value of reaching an audience, but also the value of limiting a competitor's opportunity to reach that audience. Broadcasters seeking to exploit these incentives might restrict the set of eligible advertisers (for example, to local bidders or to political advertising) in order to increase the stakes for those included.[3] Niche publications that serve a specialized audience like political staffers (The Hill, Roll Call or Politico) or the entertainment industry (Variety and The Hollywood Reporter) trade-off a loss of general advertisers (bad for revenue) for increased stakes for the remaining advertisers that compete for the finite attention of a specific audience. Search engine advertising exhibits a similar dynamic: while fewer companies might bid for specific search words, those that do are likely to be competing for the same customers. As the benefit of winning also involves the value of displacing a rival's access, firms would presumably be willing to pay more for "clicks" relative to the revenue generated on narrower searches.

The paper proceeds as follows. We first review the relevant literatures and highlight our contributions. We then present the model and consider an environment in which, despite the inherent asymmetry between caused by polarization, the bidders submit symmetric bids in equilibrium (Section 3). In this environment it is straightforward to fully characterize bidding strategies and revenue across different auction formats, and derive a general condition on polarization and the distribution of private valuations for which excluding the neutral bidder is optimal. In Section 4 we relax our symmetry conditions, but restrict our analysis to a second price auction when valuations are exponentially distributed. We show that the intuitions and trade-offs between valuation dispersion and polarization continue to hold. We further demonstrate that the welfare effects of strategic exclusion can be very high and that our results are robust to policy motivation on the part of the seller. Finally, in Section 5 we allow for arbitrary distributions of valuations and a general class of auctions and focus on strategic participation by bidders. As noted in Jehiel and Moldovanu (1996), bidders may have an in-

[2] For an example with clear externalities, consider the political delegations of Maryland, Washington and California all seeking to gain the largest possible share of the federal sea port infrastructure budget. As the port of Baltimore competes in the Atlantic trade, the competitive externality imposed on the port of Baltimore by losing to either Long Beach or Seattle is approximately the same. In contrast, for both Long Beach and Seattle losing to Baltimore is more appealing. Thus, the stakes if Long Beach is pitted directly against Seattle are higher than if Baltimore is also included.

[3] Exclusive sponsorships such as the IOC's Olympic Partners Program explicitly only allow for one sponsor in broad categories such as financial services and fast food. Such a program creates a polarized auction where firms compete for exclusive rights to advertise rather than share ad space with rivals.

centive not to participate in an auction when externalities are present. We show that the seller can induce some bidders to participate by excluding others and establish an upper bound on the seller's revenue from inviting all three bidders when she cannot compel participation. Again, the seller benefits from excluding the neutral bidder when polarization is sufficiently high. Section 6 concludes.

Related Literature

There is a large literature that models lobbying as a winner-pay competition between polarized interest groups, but it generally fails to account for the composition of the competing interest groups. Classic examples include Bernheim and Whinston (1986b), Grossman and Helpman (1996), and Besley and Coate (2001). Two explanations for the endogenous composition of interest groups are Olson (1965) and Felli and Merlo (2006). Olson argues that because the stakes are lower for moderates they have less incentive to solve the collective action problem necessary for political participation. Our results complement this logic by suggesting a further hurdle for moderates. Felli and Merlo (2006) build on Besley and Coate (2001), and model lobbying as a bargain over policy and transfers where the politician selects the lobbies from whom she will accept offers. Because the willingness of a lobbyist to pay for policy concessions depends on the politician's preferred policy position, when utility functions are strictly concave the politician bargains with lobbyists across the ideological divide and excludes the polarized lobbyist closest to her. Their model predicts that at least one polarized interest will always be excluded, the moderate will sometime be included, and that the presence of lobbyists has a moderating effect on policy. This contrasts with our results that stress the inclusion of polarized interests, the exclusion of moderates, and the polarization of policy. While we believe that the winner pay component is relevant in many parts of the lobbying process, others, including Baye et al. (1993) and Che and Gale (1998), have considered lobbying as an all-pay contest without considering the exclusion of moderates.

Our paper is part of a large literature on auctions; see Krishna (2009) for an overview. One of the general conclusions of this literature is that the seller benefits from having more bidders participating. Indeed a classic paper, Bulow and Klemperer (1996), shows that in an independent private values setting the gains from adding one additional bidder swamp the gains from auction design. The principle that reducing the number of bidders lowers revenue—referred to as the "bidder exclusion effect"—is so well established that it has emerged as a key identifying feature for empirical work on auctions (Coey, Larsen, and Sweeney, 2019).

There are, however, some previous papers that demonstrate that, as in our setting, revenue can be increased when some bidders are removed. Baye et al. (1993) is particularly relevant, showing that it can be optimal to exclude those with high valuations in an all-pay auction. The reason is that, if one bidder is too far ahead, the other bidders will bid cautiously knowing it will likely be a losing effort.[4] The incentive to bid less aggressively in an auction when disadvantaged is not present in winner-pay auctions, which are the main focus of our analysis. Moreover, in our model, interest groups do not necessarily differ in the private value they attach to a contract, but rather in their location on the ideological spectrum, incorporating the ideological component inherent in most political lobbying. In addition, our prediction that the political process will sideline moderate interests seems more empirically relevant than the prediction that it will sideline those with the most resources.

In an almost common-values auction setting, Bulow and Klemperer (2002) show that fewer bidders can increase revenue if the winner's curse is sufficiently strong. We consider a winner-pay setting in which valuations are independent and private and so there is no winner's curse effect. We demonstrate that externalities can make it possible for the seller's revenue to increase when some potential buyers are excluded, and identify the bidders it is optimal for the seller to exclude.

In addition, there is a literature on procurement auctions that considers whether greater competition can ever harm the auctioneer by increasing the amount of the lowest, winning bid. Li and Zheng (2009) consider a first-price procurement auction in which entry into the auction is endogeneous and costly. They show that, as the number of potential bidders increases, this can discourage other bidders' participation and potentially increase the winning bid. Relatedly, Chakraborty, Khalil, and Lawarree (2017) show that increased competition in a procurement auction can make it more difficult to incentivize bidders to reveal cost information which can decrease the principal's welfare.

We are not the first to consider auctions in which bidders care who they lose to. Externalities arise naturally both in political settings and in private environments when firms interact downstream (Rockett, 1990; Jehiel and Moldovanu, 2000). Auctions with identity-dependent externalities have been studied by authors including Funk (1996), Jehiel and Moldovanu (1996, 2000), Jehiel, Moldovanu, and Stacchetti (1996, 1999), and Das Varma (2002); see Jehiel and Moldovanu (2005) for a survey of this literature. While much of the literature on auctions with externalities (e.g., Jehiel et al., 1996, 1999) has focused on optimal auctions, we are interested in environments in which the seller lacks the commitment power necessary to

[4] Relatedly, Fullerton and McAfee (1999), Che and Gale (1998) and Che and Gale (2003) show that caps on expenditures or restricting entry to two bidders can increase revenues in an all-pay auction, or when there is an investment stage prior to a first-price auction.

run an optimal auction. If the seller could commit to the optimal auction, increasing the number of bidders could only increase the seller's revenue. The previous literature on standard auctions with externalities (e.g., Jehiel and Moldovanu, 1996, 2000; Das Varma, 2002) has not considered the incentive for the seller to exclude bidders. Furthermore, it has largely focused on reciprocal or cyclical externalities, which means that each bidder is ex-ante symmetric. Our interest is the inherent asymmetry between neutral and polarized bidders.

Finally, our model is related to the literature on political polarization and the influence of polarized candidates and interests. In a model of repeated elections, Van Weelden (2013) shows that a moderate voter may prefer to elect polarized candidates over moderate ones because such candidates can be motivated, by the threat of replacement by a candidate on the opposite side of the spectrum, to work harder to secure re-election. Hirsch and Shotts (2015) consider an all-pay model of competitive policy development in which legislators propose the ideological content of a bill and also invest in the quality of the legislation. They show that increasing polarization incentivizes legislators to invest more in creating high quality legislation. Klose and Kovenock (2013, 2015) also consider all-pay auctions with externalities: the former gives conditions to have only two active bidders, and the latter shows that in equilibrium the two active bidders will be the extremists, with moderates driven out of the contest. The above papers all present models of complete information in which polarized candidates are incentivized to exert more effort/resources and so moderate alternatives lose for sure in equilibrium. Consequently, removing moderates would have no effect on equilibrium outcomes. In contrast, our setting is one of incomplete information, in which each bidder wins with positive probability unless actively excluded by the seller. We identify conditions under which it is strictly revenue enhancing to actively exclude moderates from the bidding process.

2. Model

There are three bidders $i \in \{-1, 0, 1\}$ who bid over a single, indivisible contract. We assume that the seller cares only about revenue, although we relax this in Subsection 4.1. The seller is risk neutral and has a reservation value of 0 if the contract is not sold. Each bidder i receives utility

$$v_i - p_i$$

if she wins the contract and pays price p_i. If bidder i does not win the contract, her utility can depend on who does. We assume that bidder 0 is the moderate or compromise bidder: each other bidder receives 0 if she wins. We refer to her as the *neutral* bidder. However, bidder

$i \in \{-1, 1\}$ receives utility $-k$ if bidder $-i$ wins. In this sense bidders -1 and 1 are *polarized*: they would rather the contract go to the neutral bidder than to the other polarized bidder. The parameter $k > 0$ is a measure of how adversarial the interests of bidder -1 and 1 are.[5] We assume the neutral bidder receives payoff $-k/2$ if one of the polarized bidders win, and so a polarized winner imposes a disutility also on the neutral bidder, but not as large as that on the other bidder. This means that, if each of the three bidders is equally likely to win, each bidder experiences the same expected negative externality.

We consider an environment in which each bidder's valuation is drawn from a continuous distribution $v_i \sim F_i(\cdot)$ with strictly positive density $f_i(\cdot)$ on $(\underline{v}_i, \overline{v}_i)$, where $0 \leq \underline{v}_i < \overline{v}_i \leq \infty$. We assume that the seller is constrained to a second price auction, though we will relax this and demonstrate robustness to alternative auction formats in Subsection 3.2 and Section 5. While the seller cannot commit to an optimal auction, before the auction begins she decides on the set of bidders allowed to participate, which is publicly observed. Each invited bidder i submits bid $b_i \in \mathbb{R}_+$ simultaneously in a second price auction. The contract is awarded to each participating bidder who submitted a highest bid with equal probability, even if that bid is 0.[6] If only one bidder participates in the auction the contract is awarded to that bidder at price 0. There is no opportunity for resale after the contract is awarded.

Each bidder i's strategy consists of the choice of which bid to submit as a function of her own valuation and the set of bidders invited to participate. The timing of the game is as follows:

1. The seller decides on the non-empty set of bidders $B \subseteq \{-1, 0, 1\}$ to invite.
2. Each invited bidder independently realizes her valuation v_i.
3. The invited bidders simultaneously submit a bid in a second-price auction.
4. The highest bid wins the contract with the winner paying the second highest bid.

Defining

$$\tilde{v}_i := v_i - \underline{v}_i,$$

[5] In our model there is only one contract for sale and so if bidder i wins the contract then bidder $j \neq i$ is not able to purchase a similar contract from another seller. In some environments there may be a "parallel" market bidders could turn to, in which case the value of preventing the opponent from winning a particular contract would be dampened (i.e. k would be lower), and bidders would face a choice of which contract(s) to bid on. An analysis of that problem is beyond the scope of this paper.

[6] The tie-breaking rule among those participating in the auction is not important for the results; in equilibrium two or more bidders submitting the same bid will be a 0 probability event.

we can define the bidding strategy of each player, given the bidders included in the auction, as a function $b_i(\cdot)$ of $\tilde{v}_i$, the amount the bidder's valuation exceeds the lowest possible level.

We say that the distribution of valuations is **symmetric** if there exists a $\underline{v}$ and $F(\cdot)$ such $\underline{v}_{-1} = \underline{v}_1 = \underline{v}$, and $\tilde{v}_i \sim F(\cdot)$ for all $i \in \{-1, 0, 1\}$. Symmetry implies that the valuations of the polarized bidders, $i \in \{-1, 1\}$, are drawn from the same distribution. However it allows the distribution for bidder $i = 0$ to be shifted up or down relative to her rivals. We say that a symmetric auction is **strongly symmetric** if $\underline{v}_0 = \underline{v}$; given symmetry this also implies that $\overline{v}_0 = \overline{v} := \overline{v}_1 = \overline{v}_{-1}$. The strongly symmetric case then corresponds to a case in which, absent externalities, valuations would be symmetric.

We will maintain the assumption that valuations are symmetric throughout the paper, but only assume strong symmetry in Section 3. When we study the strongly symmetric case in Section 3 this condition ensures the existence of an equilibrium in which each bidder is equally likely to win. However we are also interested in cases in which the neutral bidder is advantaged or disadvantaged, and so will also consider environments that are not strongly symmetric. We solve for the Bayesian Nash Equilibrium, henceforth equilibrium,[7] in weakly undominated strategies in the bidding "subgame". We then allow the seller to choose the revenue maximizing set of bidders to invite given the bidding game.

We say that an equilibrium is **symmetric** if bidder -1 and bidder 1 use the same bidding strategies. That is, $b_1(\cdot) = b_{-1}(\cdot)$. We say that an equilibrium is **strongly symmetric** if the bidding strategy as a function of $\tilde{v}_i$ is the same for each $i \in \{-1, 0, 1\}$. That is, there exists a $b(\cdot)$ such that $b_i(\cdot) = b(\cdot)$ for $i = \{-1, 0, 1\}$. For an equilibrium to be symmetric the polarized bidders must be using the same bidding strategy, and so win with equal probability, but the neutral bidder could be using a different strategy; to be strongly symmetric all three bidders must be using the same bidding strategy and win with equal probability. We will show that a strongly symmetric equilibrium exists if the distribution of valuations is strongly symmetric.

Much of our analysis will focus on the second price auction. Without externalities, each bidder has a dominant strategy to bid her true valuation in a second price auction. In our setting, however, part of the benefit of winning the contract can be preventing another bidder from winning. When the expected negative externality from losing the contract is independent of the bid, it is an equilibrium for each bidder to bid her private value plus the expected negative externality. This holds if there is only one other bidder or if each bidder is using the same bidding strategy (Jehiel et al., 1999), which is the case when we have a strongly symmet-

[7] As there are a continuum of of valuations, in equilibrium each bidder must be optimizing almost everywhere. As is standard, we omit the almost everywhere quantifier for simplicity. However this means that in our uniqueness results the strategies will be unique up to a deviation for a measure-0 set of valuations.

ric equilibrium. Without strong symmetry, the analysis becomes more complicated and we can only characterize the bidding strategies under certain distributional assumptions; we will also present results that do not require a complete characterization of the bidding strategies. Throughout our analysis we assume that the seller is able to commit to a set of bidders to allow to participate but cannot commit not to sell to the highest bidder.

Before proceeding to the equilibrium analysis, we define $\tilde{v}(j,n)$ to be the j-th highest order statistic of n independent draws from distribution $F(\cdot)$. In what follows we are interested in comparing the seller's revenue and the efficiency of the equilibrium with all three bidders participating to having only two bidders. In Section 3 we calculate the equilibrium strategies, seller revenue, and welfare with two polarized bidders, and with three bidders under strong symmetry. In Section 4 we relax the strong symmetry assumption and allow the neutral bidder to either be advantaged or disadvantaged; for those results we consider private valuations that are exponentially distributed. Finally, in Section 5, we show that the main conclusions extend for general distributional assumptions and many different auction formats when bidders can strategically choose whether to participate in the auction.

3. Results with Strong Symmetry

3.1. Two Symmetric, Polarized Bidders

We first characterize the bidding strategies when the neutral bidder ($i = 0$) is prevented from participating. As each polarized bidder $i \in \{-1, 1\}$ then knows that bidder $-i$ will win if they don't, this is a symmetric independent private values auction where each bidder $i \in \{-1, 1\}$ has net valuation $k + v_i = k + \underline{v} + \tilde{v}_i$ with $\tilde{v}_i \sim F(\cdot)$. It is then straightforward to characterize the unique equilibrium in weakly undominated strategies. (The proofs of all results are included in the Appendix.)

Proposition 1. *In the unique equilibrium in weakly undominated strategies each bidder bids $k + \underline{v} + \tilde{v}_i$ and the seller's (expected) revenue is*

$$k + \underline{v} + \mathbb{E}[\tilde{v}(2, 2)]. \tag{1}$$

We now compare the seller's revenue when the neutral bidder is prevented from participating to the case in which all three bidders can submit bids.

3.2. Three Bidders with Strong Symmetry

Suppose there are three bidders with a strongly symmetric distribution of valuations. That is $\underline{v}_0 = \underline{v}$ and $\tilde{v}_i \sim F(\cdot)$ for all $i \in \{-1, 0, 1\}$. Note that, if each bidder believes that, conditional on not winning the contract, each of the other two bidders are equally likely to win, then they all have the same expected net benefit of winning the contract for any $\tilde{v}_i$: each bidder is willing to pay $\frac{k}{2} + \underline{v} + \tilde{v}_i$. A strongly symmetric equilibrium then exists in which each bidder bids her net valuation conditional on each other bidder winning with equal probability.[8]

Proposition 2. *In the unique strongly symmetric equilibrium each bidder bids $\frac{k}{2} + \underline{v} + \tilde{v}_i$. The seller's revenue is*

$$\frac{k}{2} + \underline{v} + \mathbb{E}[\tilde{v}(2, 3)]. \tag{2}$$

An immediate Corollary of Proposition 1 and Proposition 2 is that preventing the neutral bidder from participating is revenue enhancing if and only if the difference between the expectation of the second order statistics with two and three draws from $F(\cdot)$ is not too large.

Corollary 1. *Conditional on the strongly symmetric equilibrium, the seller's revenue is higher from preventing the neutral bidder from participating if*

$$\mathbb{E}[\tilde{v}(2, 3)] - \mathbb{E}[\tilde{v}(2, 2)] < \frac{k}{2},$$

and higher from allowing the neutral bidder to participate if

$$\mathbb{E}[\tilde{v}(2, 3)] - \mathbb{E}[\tilde{v}(2, 2)] > \frac{k}{2}.$$

This corollary says that it is revenue enhancing to exclude the neutral bidder if and only if the second order statistic of three private value draws is not too much larger than the second order statistic from two private value draws. Roughly speaking, the expected difference between the order statistics is small when bidders' valuations are compressed. When there is little variation in bidders' private valuations, the *competition effect* is muted and so the *stakes effect* dominates.

How big the difference in the order statistics must be depends on k, with a greater degree of polarization increasing the range of distributions for which it is profitable to exclude the neutral bidder. This is because, as k gets larger, the cost of losing to a bidder on the opposite

[8] Equilibria that are not strongly symmetric may also exist depending on parameters.

side increases until the *stakes effect* begins to dominate. An alternative way to view Corollary 1 is that, for any $F(\cdot)$, it will be profitable to exclude the neutral bidder if and only if k is sufficiently large. As $k \to 0$ the model collapses to an independent private values auction without externalities, and so it can never be profitable to exclude the neutral bidder.

We illustrate the conditions from Corollary 1 in the following two examples where bidder valuations are drawn from the uniform distribution and the exponential distribution respectively. Both examples illustrate that greater variability in the valuations makes it less likely that the seller benefits from excluding the neutral bidder.

Example 1. Suppose v_i are uniformly distributed on $[\underline{v}, \overline{v}]$. Then $F(\cdot)$ is the uniform distribution on $(0, \overline{v} - \underline{v})$, and so $\mathbb{E}[\tilde{v}(2,2)] = \frac{\overline{v}-\underline{v}}{3}$ and $\mathbb{E}[\tilde{v}(2,3)] = \frac{\overline{v}-\underline{v}}{2}$. Hence, by Corollary 1, the seller's revenue is higher from preventing the neutral bidder from participating if $k > \frac{\overline{v}-\underline{v}}{3}$ and higher from allowing the neutral bidder to participate if $k < \frac{\overline{v}-\underline{v}}{3}$. ‖

Example 2. Suppose $F_i(\cdot)$ is the exponential distribution on $(\underline{v}, \infty)$ with rate parameter $\lambda > 0$. Then the density of each bidder's $\tilde{v}_i$ is

$$f(\tilde{v}_i) = \lambda \exp^{-\lambda \tilde{v}_i}.$$

For the exponential distribution it is straightforward to calculate that

$$\mathbb{E}[\tilde{v}(2,2)] = \frac{1}{2\lambda}, \tag{3}$$

and

$$\mathbb{E}[\tilde{v}(2,3)] = \frac{5}{6\lambda}. \tag{4}$$

Therefore,

$$\mathbb{E}[\tilde{v}(2,3)] - \mathbb{E}[\tilde{v}(2,2)] = \frac{1}{3\lambda} < \frac{k}{2},$$

if and only if

$$\lambda > \frac{2}{3k}.$$

With an exponential distribution, the higher λ, the lower the mean and variance of the distribution of values, so again we see that it is profitable to prevent participation by the neutral bidder if and only if the distribution of valuations is sufficiently concentrated. ‖

The comparison in Corollary 1 generates a simple, clean prediction of when the seller benefits, and when the seller is harmed, by excluding the neutral bidder. However, it also raises a number of additional questions. One question is whether or not the conclusion is specific to the second price auction. This is important because many environments of interest (such as interest group pledges or offers from acquiring firms) would more closely resemble a first price auction than a second; similarly, many lobbying environments have a strong all-pay component. Conditional on a strongly symmetric equilibrium, or a symmetric equilibrium with two bidders, this is straightforward because the revenue equivalence theorem applies.

Proposition 3. *Suppose the contract is allocated through a first price or an all-pay auction. Then,*

1. *if $B = \{-1, 1\}$ and the distribution of valuations is symmetric, in the unique symmetric equilibrium the seller's revenue is the same as in Proposition 1.*
2. *if $B = \{-1, 0, 1\}$ and the distribution of valuations is strongly symmetric, in the unique strongly symmetric equilibrium the seller's revenue is the same as in Proposition 2.*

For strongly symmetric valuations then the revenue comparison between including and excluding the neutral bidder, if the contract is allocated by either a first price or all-pay auction, is the same as in Corollary 1. Our results on the benefit of excluding the neutral bidder then apply for different standard auctions under strong symmetry.

Our analysis so far assumes that the environment and the equilibrium are strongly symmetric. If the bidders' valuations are not strongly symmetric, a strongly symmetric equilibrium will not exist. Due to the technical difficulties in analyzing asymmetric auctions, we are not able to provide a full characterization of bidding strategies and welfare for general distributions without strong symmetry. However we demonstrate that strong symmetry is not essential for our conclusions in two ways. In Section 4 we fully characterize a broader class of equilibria relaxing the assumption of strong symmetry in a second price auction when private valuations are exponentially distributed. Here, although there is an inherent asymmetry in the bidders, the memoryless property still allows for a characterization of a class of equilibrium strategies. For more general distributions and auction formats a characterization of bidding strategies and seller revenue is more complicated. However, when the seller cannot compel participation, we can construct a bound on the seller's revenue that holds across different auction formats and distributions of private valuations. We take this approach in Section 5. First we pause to discuss the welfare consequences of excluding bidders.

3.3. Welfare

We have found conditions under which it is beneficial, from the perspective of the seller, to prevent the neutral bidder from participating. However, removing the neutral bidder creates inefficiencies: the neutral bidder may have the highest valuation **and** the neutral bidder imposes the smallest negative externality on other bidders by winning the contract. So even though excluding the neutral bidder can be revenue enhancing, it is inefficient for society.

To make this precise, note that the total negative externality from selling to bidder $i \in \{-1, 1\}$ rather than 0 is $3k/2$ higher. So it is efficient to sell to the neutral bidder unless the valuation of one of the polarized bidders is at least $3k/2$ higher than the neutral bidder's valuation. This is the only efficient way for the contract to be allocated: there would otherwise exist a set of transfers and a way to reallocate the contract that would result in a pareto improvement.

Remark 1. In the efficient allocation the contract should be awarded to bidder 0 if and only if $v_0 \geq \max\{v_{-1}, v_1\} - 3k/2$. If $v_0 < \max\{v_{-1}, v_1\} - 3k/2$ then it is efficient to award the contract to the bidder $i \in \{-1, 1\}$ with the higher valuation.

The equilibrium allocation will not necessarily be efficient even if all bidders are allowed to participate. In particular, the neutral bidder will win too infrequently.[9] This is because, in equilibrium, the bidder with the highest private valuation will win the contract, but from an efficiency standpoint the neutral bidder should sometimes be awarded the contract even when her valuation is lower than one of the polarized bidders because she does not impose a negative externality on other bidders. Thus, in equilibrium, the neutral bidder wins only in a proper subset of the cases for which it is welfare maximizing for her to win, and excluding the neutral bidder always reduces the efficiency of the auction mechanism. It is not necessarily true, however, that efficiency is always enhanced by including more bidders. When the probability that $v_i > v_0 + 3k/2$ for $i \neq 0$ is low, aggregate welfare—defined as the expected sum of utilities for the seller and the three bidders—can be higher if the polarized bidders are excluded and the neutral bidder always receives the contract. However, although it could increase aggregate welfare, the seller would never benefit from excluding polarized bidders.

Remark 2. With all bidders included, the equilibrium allocation may not be efficient. Preventing the neutral bidder from participating could increase or decrease seller revenue but always decreases aggregate welfare. Preventing a polarized bidder from participating could increase or decrease aggregate welfare but always reduces the seller's revenue.

[9] That standard auctions with externalities are not necessarily efficient has been recognized in the previous literature. See, for example, Jehiel and Moldovanu (2005).

4. Symmetric Bidders and The Exponential Distribution

We now consider a symmetric setting, but one that is not necessarily strongly symmetric, and will show that strong symmetry is not necessary to generate the revenue comparisons in Section 3. In addition, by relaxing the strong symmetry assumption we can consider environments in which the neutral bidder is more or less likely to win than a polarized bidder.

In order to make the problem tractable we make a specific functional form assumption and assume that v_i is drawn from the exponential distribution on $(\underline{v}_i, \infty)$ with rate $\lambda > 0$ for $i \in \{-1, 0, 1\}$. Furthermore, we restrict attention to a second price auction for this section.[10] While first-price and all-pay auctions may be more common in practice, second price auctions are used in some environments in which externalities are likely to be present, such as for Google Adwords.[11]

When the distribution of valuations are not strongly symmetric a strongly symmetric equilibrium will not exist; rather, we focus on a weaker selection, looking at equilibria that are **anonymous**. We say that an equilibrium is anonymous if the bidding strategies are such that, for each i, the probability of each other bidder in B winning the contract, conditional on i not winning it, is independent of i's bid, b_i. Clearly every equilibrium is anonymous if $|B| = 2$. When $B = \{-1, 0, 1\}$, an equilibrium is anonymous if, for all distinct i, j, and k,

$$Pr(b_j(\tilde{v}_j) > b_k(\tilde{v}_k) | \max\{b_j(\tilde{v}_j), b_k(\tilde{v}_k)\} > b_i)$$

is constant in b_i. It follows that every strongly symmetric equilibrium is anonymous and every anonymous equilibrium is symmetric. Anonymous equilibria have the property that the expected negative externality imposed on a bidder is independent of her bid, which greatly simplifies the equilibrium characterization. Hence, in a second price auction, it will be an equilibrium for each bidder to bid their own private valuation plus the expected negative externality if they do not win the auction. The memoryless property of the exponential distribution allows us to characterize the bidding strategies, and equilibrium revenue, even when $\underline{v}_0$ is larger or smaller than $\underline{v}$. An anonymous equilibrium always exists with symmetric bidders and exponentially distributed valuations. We have already solved for the strongly symmetric equilibrium when $\underline{v}_0 = \underline{v}$ in Example 2, which may or may not be the unique anonymous

[10] Unlike in Section 3, the revenue equivalence theorem does not hold in this environment, so the incentive to exclude the neutral bidder under a first-price or all-pay auction would be different than under a second-price auction.

[11] However, we assume there is only one contract and so no second position, unlike in many auctions for online advertisements.

equilibrium when depending on the parameters. When $\underline{v}_0 \neq \underline{v}$ a strongly symmetric equilibrium will not exist, and there may or may not be a unique anonymous equilibrium. As polarized bidders value winning the contract more if they are more likely to lose to the other polarized bidder, multiple anonymous equilibria can exist: those in which polarized bidders expect each other to bid more and less aggressively. This multiplicity arises when λ and k are sufficiently large, and so the negative externality is sufficiently important relative to the private valuation. As this is when excluding the neutral bidder is most advantageous there exists a cutoff for when excluding the neutral bidder is beneficial that holds across all anonymous equilibria.

Proposition 4. *Suppose $\underline{v} \in (\underline{v}_0 - k/2, \infty)$. Then an anonymous equilibrium exists for all $\lambda > 0$ when the seller includes all three bidders. In every anonymous equilibrium, there exists an $\alpha \in (0, 1)$ such that bidder 0 wins if and only if $v_0 \geq \max\{v_{-1}, v_1\} + (\alpha - 1/2)k$. Otherwise the polarized bidder with the higher valuation wins.*

Proposition 4 shows that, when the distribution of valuations is exponential, an anonymous equilibrium exists and takes a simple form. Anonymous equilibria are characterized in terms of α, the relative probability of a polarized rather than neutral bidder winning the contract. Modulo the bids of sure losers, the neutral bidder bids her valuation plus the negative externality if she does not win, $b_0(\tilde{v}_0) = \underline{v}_0 + k/2 + \tilde{v}_0$. The polarized bidders bid their valuation plus the expected negative externality if they don't win, $b_i(\tilde{v}_i) = \underline{v} + \alpha k + \tilde{v}_i$. The details of the bidding strategies, including the bids of sure losers and characterization of α, are provided in the Appendix.

Given the equilibrium characterization, our next proposition shows that a version of the result in Corollary 1 holds: it is profitable to exclude the neutral bidder if and only if the private valuations are sufficiently compressed.

Proposition 5. *Suppose $\underline{v} \in (\underline{v}_0 - k/2, \infty)$. Then, there exists a $\overline{\lambda} > 0$ such that, if $\lambda > \overline{\lambda}$, the seller's revenue is higher in the equilibrium with two polarized bidders than in any anonymous equilibrium with all three bidders included. If $\lambda < \overline{\lambda}$ there exists an anonymous equilibrium with all three bidders included that generates higher revenue than with two polarized bidders.*

As noted in Subsection 3.3, excluding the neutral bidder is harmful in terms of aggregate welfare. The next proposition shows how large the distortion can be when the seller prevents the neutral bidder from participating: there exist parameters for which the neutral bidder would win with probability close to one if all three bidders were invited, yet the seller optimally prevents the neutral bidder from participating.

As there can exist multiple anonymous equilibria, the welfare comparison depends on which equilibrium is selected. We focus on a selection of equilibria in which $b_i(\tilde{v}_i)$ is continuous in λ for all $i \in \{-1, 0, 1\}$ and $\tilde{v}_i$. Then, as λ gets large, the probability of the neutral bidder winning gets arbitrarily close to one when $\underline{v} < \underline{v}_0$. However, if $\underline{v} > \underline{v}_0 - k/2$ it is profitable to exclude the neutral bidder.

Proposition 6. *Suppose that $\underline{v} \in (\underline{v}_0 - k/2, \underline{v}_0)$, and consider a continuous selection of anonymous equilibria. Then, for any $\varepsilon > 0$, there exists $\lambda^*(\varepsilon)$ such that, for all $\lambda > \lambda^*(\varepsilon)$,*

1. *the neutral bidder winning the contract is efficient with probability greater than $1 - \varepsilon$.*
2. *if all three bidders are included the neutral bidder wins, and the final allocation is efficient, with probability greater than $1 - \varepsilon$.*
3. *the seller will exclude the neutral bidder.*

So we see that excluding the neutral bidder, while revenue enhancing, is always bad for welfare. In fact, Proposition 6 identifies cases in which it is optimal for the seller to prevent participation by the neutral bidder, even though the equilibrium allocation with three bidders is efficient with probability arbitrarily close to one, and the equilibrium allocation with the neutral bidder excluded is inefficient with probability arbitrarily close to one.

In order to get the closed form revenue results of this section it has been important to focus on an environment in which the equilibrium bids have the memoryless property. When valuations are not exponentially distributed, or if the contract is allocated by a first-price or all-pay auction, the memoryless property would no longer hold. As we would then be considering a general asymmetric auction, it would not be possible to solve for the seller's revenue explicitly. Still, we expect similar tradeoffs to hold. We demonstrate this in Section 5, which provides more general conditions under which excluding the neutral bidder is revenue enhancing for different auction formats.

4.1. Policy-Motivated Seller

We now consider the possibility that the seller may also care about who wins the contract. In order to bias in favor of the neutral bidder, we assume the seller is ideologically aligned with the neutral bidder, and so prefers her to win. In particular, we assume the seller receives a disutility of $c > 0$ if she sells to one of the polarized bidders. In order to take into account her own policy preferences, we assume the seller uses an augmented second-price auction: she sells to the bidder who submits the highest bid, but if this is a polarized bidder, they must pay the second highest bid plus c. This means that a polarized bidder must compensate the seller for imposing a negative externality on her. As the polarized bidders then have an incentive to reduce their bid by c, this auction is equivalent to one in which each polarized bidder's valuation is exponentially drawn from $(\underline{v} - c, \infty)$ and the seller does not receive a disutility from selling to a polarized buyer. We can then apply our previous results to determine the seller's payoff. When k is sufficiently large and the uncertainty about private valuations is small, the seller benefits from excluding the neutral bidder from the auction.

Proposition 7. *Suppose $\underline{v} - c > \underline{v}_0 - k/2$. Then there exists a $\overline{\lambda}$ such that the seller benefits from excluding the neutral bidder if $\lambda > \overline{\lambda}$.*

This result shows that adding a disutility to the seller from a polarized bidder winning may reduce, but does not eliminate, the range of parameters for which excluding the neutral bidder is revenue enhancing.

5. General Results with Strategic Bidder Participation

So far we have assumed that all invited bidders participate in the auction. However, as Jehiel and Moldovanu (1996) demonstrate, it is possible, in the presence externalities, that bidders may have a strategic incentive not to participate even if doing so is costless. In this section we extend that insight by noting that the seller can change the incentives to participate by altering the set of invited bidders. We then look for the revenue maximizing set of bidders to invite when the bidders have the option not to accept the invitation.

We assume that the seller invites the bidders first. Then, after observing the set of bidders who were invited, each invited bidder simultaneously decides whether or not to attend.[12] Finally, the attending bidders submit bids after observing which bidder(s) accepted the invitation. We assume that all bidders receive a non-positive payoff if no bidders attend and the contract is not allocated. Bidders only learn their v_i if they attend, and there are no costs of attending the auction.[13] The distribution of valuations is symmetric, but not necessarily strongly symmetric.

[12] This timing of moves ensures that a bidder can't change which other bidders are invited by deciding not to participate, something that would open up additional strategic considerations.

[13] We could allow a positive cost of attending. While our results would need to be adapted slightly, the set of bidders who would attend the auction would be unchanged if this cost were sufficiently small.

The incentive for strategic non-participation can only emerge if the bidder wants to influence which other included bidder wins the contract. Consequently the neutral bidder will always participate if permitted to do so, and if only two bidders are invited both will attend. However, if all bidders are invited, it is not always the case that both polarized bidders will attend, for exactly the reasons identified in Jehiel and Moldovanu (1996). By not attending a polarized bidder can reduce the other polarized bidder's willingness to pay, making it more likely the neutral bidder wins. Of course, if one of the polarized bidders doesn't attend the auction, this can only lower the seller's revenue from allowing all three bidders to participate.

Remark 3. Suppose the seller invites bidders $B \subseteq \{-1, 0, 1\}$. Then, in any auction format in which the high bid wins and the bidder's payment cannot exceed her bid, not attending the auction is a weakly dominated strategy for (a) bidder 0 for any B with $0 \in B$; (b) both invited bidders if $|B| = 2$.

The above remark demonstrates that, although it is possible that bidders may not attend if invited, this possibility can only increase the incentive to exclude the neutral bidder. Excluding the neutral bidder ensures the polarized bidders attend, whereas it is possible they may not when the neutral bidder is included. As such, the fact that the seller cannot compel participation means that excluding the neutral bidder will be beneficial for a broader range of parameters.

This approach allows us to create a bound on revenue that holds for general distributions and auction formats, at least when the private value component is not too important. The key is to look at what happens when a polarized bidder doesn't attend the auction when all three bidders are invited. Then, since the neutral bidder will always attend by Remark 3, the resulting auction will involve the neutral bidder and at most one polarized bidder. If the neutral bidder would defeat the polarized bidder with high probability this creates a lower bound on each polarized bidder's payoff from not attending. As this lower bound on payoffs is close to 0, a polarized bidder will not attend the auction and bid more than their private valuation in order to prevent the bidder on the other side from winning. This, in turn, generates a bound on the seller's revenue.

We now consider any distribution of bidder private valuations and a broader class of auction mechanisms. We say that an auction satisfies condition $\mathcal{FS}$ if the auction is strategically equivalent to either the first or second price auction when there are only two bidders. Many different auction formats that could differ with three bidders satisfy $\mathcal{FS}$, including a second price auction, a first price auction, and an English auction.[14] Condition $\mathcal{FS}$ is also satisfied

[14] With more than two bidders and externalities, the second price and English auction are not necessarily

by a combinatorial auction in which bidders can subsidize others to influence who wins the contract if they don't, as once there are only two bidders remaining there is nobody left for a bidder to subsidize. Allowing for such auctions is important since, if the bidding is to influence policy, there could be coalitions formed between different interested bidders to push for the same policy.[15] Such an auction would satisfy $\mathcal{FS}$ and so our results on when it would be optimal to exclude the neutral bidder apply to combinatorial auctions as well.

What is ruled out by $\mathcal{FS}$ are auction formats that rely on the seller being able to commit to sell to the polarized bidder over the neutral bidder even if the polarized's bid is lower.[16] As discussed above, we view such commitment as implausible in many environments of interest.

We now look for equilibria in the game with bidder entry for any set B of invited bidders. We define an equilibrium in this setting to be a profile of strategies such that: (1) a Bayes Nash Equilibrium is played in the bidding stage, (2) each bidder's entry decision is a best response given the other bidders' entry decision and the play in the bidding stage. We restrict attention to equilibria in which each bidder plays a weakly undominated strategy.

The next proposition provides an upper bound on the seller's revenue for any equilibrium under any auction that satisfies $\mathcal{FS}$. Note that we are not characterizing all mechanisms that satisfy this condition, or guaranteeing the existence of an equilibrium for general auction mechanisms. Rather we are constructing an upper bound on the seller's revenue in any equilibrium of any auction mechanism that reduces to either a first or second price auction with only two bidders. In order to apply results on first price auctions from Maskin and Riley (2000, 2003) we add the additional assumption that valuations are bounded.

Lemma 1. *Suppose $\underline{v} - k/2 < \underline{v}_0$ and $\overline{v}_i < \infty$ for all $i \in \{-1, 0, 1\}$. Suppose also that the contract will be allocated by an auction that satisfies $\mathcal{FS}$. Then, for any $\varepsilon > 0$, there exists $\overline{\delta} > 0$ such that, for any twice continuously differentiable $F(\cdot)$ with $\mathbb{E}[\tilde{v}_i] < \overline{\delta}$, the seller's revenue from $B = \{-1, 0, 1\}$ is less than $\underline{v}_0 + k/2 + \varepsilon$ in any equilibrium.*

Lemma 1 applies for any distribution of valuations and any auction format that satisfies $\mathcal{FS}$. The key is that, if only the neutral and one polarized bidder attend the auction, the neutral bidder wins with very high probability. Consequently a polarized bidder can only

equivalent (Das Varma, 2002).

[15] This is closely related to the common agency literature (Bernheim and Whinston, 1986a; Dixit, Grossman, and Helpman, 1997) in which multiple principals/interest groups seek to influence one agent/policymaker through contributions.

[16] For example, if the seller could commit to a rule that if bidder $i \in \{-1, 1\}$ did not attend the auction, the contract would be given to bidder $-i$ for free, the seller could, in essence, compel participation by the bidders. Such a mechanism would not satisfy condition $\mathcal{FS}$.

be induced to participate in an auction in which, if they participate, either paying more than their private valuation or having the other polarized bidder win happens with a very low probability. This creates a bound on the seller's revenue, and sufficient conditions for the seller to benefit from excluding the neutral bidder. Our main result of this section then follows immediately from Lemma 1.

Proposition 8. *Suppose $\underline{v} - k/2 < \underline{v}_0 < \underline{v} + k/2$ and $\overline{v}_i < \infty$ for all i. Suppose also that the contract will be allocated by an auction that satisfies $\mathcal{FS}$. Then there exists $\overline{\delta} > 0$ such that, for any twice continuously differentiable $F(\cdot)$ with $\mathbb{E}[\tilde{v}_i] < \overline{\delta}$, the seller's revenue is higher in the unique symmetric equilibrium with $B = \{-1, 1\}$ than in any equilibrium with $B = \{-1, 0, 1\}$.*

This demonstrates that, for a broad class of mechanisms and distributions, when the private component is small relative to k, the seller rationally excludes the neutral bidder from participating. This means that it is optimal to exclude the neutral bidder whenever private valuations are sufficiently compressed *or* polarization is sufficiently large. Note that this conclusion is independent of any issues related to equilibrium selection: the equilibrium excluding the neutral bidder is better than in any equilibrium when all bidders are invited.

6. Conclusions

We have considered the problem of a seller who must decide whom to accept offers from given a pool of competing interests. We have identified conditions under which the seller benefits from restricting participation to those whose interests are diametrically opposed and rejecting moderate or compromise alternatives. This possibility emerges even if the seller prefers moderate outcomes, and is robust to different auction mechanisms and distributional assumptions. While the seller may benefit from excluding compromise alternatives, it can only have deleterious welfare consequences.

These results speak to the advantages of playing opposing interests against each other and generates insights about what happens when polarized interests compete. For example, teams from the same division or league competing for star talent are more polarized than those from different divisions. Consequently, we would expect that player contracts emanating from polarized competition to be overgenerous relative to fundamentals.[17] Similar competitive

[17] For example, in 2015 free-agent NFL running back DeMarco Murray's choice came down to the Dallas Cowboys and Philadelphia Eagles, rivals in the NFC East. Murray signed an enormous contract with the Eagles described as a "double victory for the Philadelphia Eagles" because it "weakened the toughest competition in the NFC East in the process" (Bell, 2015) but his subsequent production in Philadelphia was far below that of other players with similar contracts.

effects arise when technologies exhibit strong network effects. In such settings, sellers have an incentive to limit the pool of potential buyers to those who stand to lose if a strong competitor won, and the price that emerges from polarized negations could well be higher the value assigned to the asset by any bidder.

Our results may also relate to the question of why the political debate appears to be dominated by the extremes. Often, this polarized debate and lack of compromise is decried as a failure of leadership or blamed on a lack of organization by moderates. However compromise, though socially efficient, may not be in the interest of sellers: rather there is an incentive to increase polarization in order to raise the stakes and increase the rents that can be extracted. In fact, the temptation to exclude moderates is highest precisely when interests are most polarized and compromise is most desirable. Viewed in this light, a lack of compromise is simply the result of strategic leaders following their incentives.

An implication is that compromise will not simply arise from a change of leadership or attitude among politicians, but rather may require different institutional arrangements. Plans that seek to fund and represent moderate views without changing incentives to exclude them may fail to produce a moderate outcome. This raises important issues in the design of mechanisms and institutions to reduce the incentives for agenda setters to exacerbate polarization. Exploring such approaches is an important avenue for future research.

A. Appendix: Proof of Results

A.1. Proofs for Section 3

Proposition 1 and Proposition 2 follow from Proposition 2.2 of Krishna (2009). Proposition 3 is immediate from Proposition 3.1 of Krishna (2009).

A.2. Detailed Characterization and Proofs for Section 4

We now characterize the set of anonymous equilibria. We will identify the conditions under which there is a unique anonymous equilibrium, and the conditions under which the revenue is higher with two bidders than in any anonymous equilibrium with three bidders. We then use these stronger results to prove Proposition 4, Proposition 5, Proposition 6, and Proposition 7. We first prove the following simple lemma.

Lemma A.1. *In every anonymous equilibrium in weakly undominated strategies there exists an* α *such that* $b_0(\tilde{v}_0) = \underline{v}_0 + k/2 + \tilde{v}_0$ *and* $b_i(\tilde{v}_i) = \underline{v} + \alpha k + \tilde{v}_i$ *for all* $\tilde{v}_i > \underline{v}_0 - \underline{v} - (\alpha - 1/2)k$ *and* $i \in \{-1, 1\}$.

Proof. By weak dominance bidder 0 must bid $\underline{v}_0 + k/2 + \tilde{v}_0$. In any anonymous equilibrium $b_{-1}(\cdot) = b_1(\cdot)$ and there exists some $\alpha \in [0, 1]$ such that

$$Pr(b_{-1} > b_0 | \max\{b_{-1}, b_0\} > b_1) = \alpha$$

for all b_1. This implies that, for any b_1, bidder 1's expected utility if she does not win is $-\alpha k$, and so 1's best response is to bid $\alpha k + v_1 = \underline{v} + \alpha k + \tilde{v}_1$. Note that this is the unique best response if $\underline{v} + \alpha k + \tilde{v}_1 > \underline{v}_0$ but if $\underline{v} + \alpha k + \tilde{v}_1 < \underline{v}_0 + k/2$ any bid in $[\underline{v} + \tilde{v}_1, \underline{v}_0 + k/2]$ is a weakly undominated best response. By symmetry, bidder -1 must then bid $\underline{v} + \alpha k + \tilde{v}_{-1}$ when $\tilde{v}_{-1} > \underline{v}_0 - \underline{v} - (\alpha - 1/2)k$. $\square$

By Lemma A.1, characterizing the anonymous equilibria consists of solving for α, the conditional probability, from the perspective a polarized bidder, that the other polarized bidder wins the auction if they do not. Our next supporting lemma provides conditions on what α can be in equilibrium when λ and k are not too large.

Lemma A.2. *Suppose* $\lambda k \leq 2$. *Then,*

1. *If* $\underline{v}_0 \leq \underline{v}$ *then there cannot exist an anonymous equilibrium with* $\alpha < 1/2$.
2. *If* $\underline{v}_0 \geq \underline{v}$ *then there cannot exist an anonymous equilibrium with* $\alpha > 1/2$.

Proof. Given Lemma A.1, in any anonymous equilibrium b_0 is exponentially distributed on $(\underline{v}_0 + k/2, \infty)$ and (up to the bids of sure losers) b_1 and b_{-1} are exponentially distributed on $(\underline{v} + \alpha k, \infty)$. This implies that in any equilibrium with $\alpha > 1/2$ must involve $\underline{v} + (\alpha - 1/2)k > \underline{v}_0$, because otherwise the neutral bidder would be winning at least as often as the polarized bidder. Similarly, any equilibrium with $\alpha < 1/2$ we must have $\underline{v} + (\alpha - 1/2)k < \underline{v}_0$. We now prove parts 1 and 2 of the proposition separately, proceeding by contradiction.

Part 1: Suppose we have an equilibrium with $\alpha < 1/2$ when $\underline{v}_0 \leq \underline{v}$. As $\underline{v} + (\alpha - 1/2)k < \underline{v}_0$ and $b_{-1} = \underline{v} + \alpha k + \tilde{v}_{-1}$ we see that, if $\tilde{v}_{-1} < \underline{v}_0 - \underline{v} + (1/2 - \alpha)k$, bidder -1 loses for sure. However, by the memoryless principle of the exponential distribution, conditional on $\tilde{v}_{-1}$ exceeding $\underline{v}_0 - \underline{v} + (1/2 - \alpha)k$ bidders -1 and 0 are equally likely to win. Therefore α must solve

$$
\begin{aligned}
\alpha &= Pr(b_{-1} > b_0 | \max\{b_{-1}, b_0\} > b_1) \\
&= \frac{1}{2}(1 - F(\underline{v}_0 - \underline{v} + (1/2 - \alpha)k)) \\
&= \frac{e^{-\lambda(\underline{v}_0 + (1/2-\alpha)k - \underline{v})}}{2}.
\end{aligned}
$$

Defining

$$
g(\alpha) := \alpha - \frac{e^{-\lambda(\underline{v}_0 + (1/2-\alpha)k - \underline{v})}}{2}, \tag{5}
$$

to have an equilibrium it must be that $g(\alpha) = 0$ for some $\alpha \in [0, 1/2)$. We now show that $g(\alpha) < 0$ for all $\alpha \in [0, 1/2)$.

Note first that as $\underline{v}_0 \leq \underline{v}$ it follows that

$$
g(0) = -\frac{e^{-\lambda(\underline{v}_0 + k/2 - \underline{v})}}{2} < 0,
$$

and

$$
g(1/2) = \frac{1 - e^{-\lambda(\underline{v}_0 - \underline{v})}}{2} \leq 0.
$$

Furthermore, differentiating $g(\cdot)$ we have that

$$
g'(\alpha) = 1 - \frac{\lambda k}{2} e^{-\lambda(\underline{v}_0 + (1/2-\alpha)k - \underline{v})}, \tag{6}
$$

and

$$
g''(\alpha) = -\frac{\lambda^2 k^2}{2} e^{-\lambda(\underline{v}_0 + (1/2-\alpha)k - \underline{v})} < 0.
$$

As this implies that $g(\cdot)$ is concave it is sufficient to show that $g(\alpha) < 0$ for any $\alpha \in [0, 1/2)$ with $g'(\alpha) = 0$. This follows because, by Equation 5 and Equation 6, $g'(\alpha) = 0$ implies that

$$
g(\alpha) = \alpha - \frac{1}{\lambda k}.
$$

Hence, when $\alpha < 1/2$, and $\lambda k \leq 2$,

$$
g(\alpha) < 0.
$$

Part 2: Suppose we have an equilibrium with with $\alpha > 1/2$ when $\underline{v}_0 \geq \underline{v}$. As $\underline{v} + (\alpha - 1/2)k > \underline{v}_0$ and $b_{-1} = \underline{v} + \alpha k + \tilde{v}_{-1}$ we see that, if $\tilde{v}_0 < \underline{v} + (\alpha - 1/2)k - \underline{v}_0$, bidder 0 loses for sure. However, by the memoryless principle of the exponential distribution, conditional on

$\tilde{v}_0$ exceeding $\underline{v} + (\alpha - 1/2)k - \underline{v}_0$ bidders -1 and 0 are equally likely to win. Therefore α must solve

$$\begin{aligned}\alpha &= Pr(b_{-1} > b_0 | \max\{b_{-1}, b_0\} > b_1) \\ &= F((\alpha - 1/2)k + \underline{v} - \underline{v}_0) + \frac{1}{2}(1 - F((\alpha - 1/2)k + \underline{v} - \underline{v}_0)) \\ &= \frac{2 - e^{-\lambda((\alpha-1/2)k+\underline{v}-\underline{v}_0)}}{2}.\end{aligned}$$

Defining

$$g(\alpha) = \alpha + \frac{e^{-\lambda((\alpha-1/2)k+\underline{v}-\underline{v}_0)}}{2} - 1, \tag{7}$$

any such equilibrium must involve $g(\alpha) = 0$. We now show that $g(\alpha) > 0$ for all $\alpha \in (1/2, 1]$. Note that as $\underline{v}_0 \geq \underline{v}$,

$$g(1) = \frac{e^{-\lambda(k/2+\underline{v}-\underline{v}_0)}}{2} > 0,$$

and

$$g(1/2) = \frac{e^{-\lambda(\underline{v}-\underline{v}_0)} - 1}{2} \geq 0.$$

Moreover,

$$g'(\alpha) = 1 - \lambda k \frac{e^{-\lambda((\alpha-1/2)k+\underline{v}-\underline{v}_0)}}{2}, \tag{8}$$

and

$$g''(\alpha) = \lambda^2 k^2 \frac{e^{-\lambda((\alpha-1/2)k+\underline{v}-\underline{v}_0)}}{2} > 0.$$

As $g(\cdot)$ is convex it is sufficient to show that $g(\alpha) > 0$ for any $\alpha > 1/2$ with $g'(\alpha) = 0$. This follows because, by Equation 7 and Equation 8, when $\lambda k \leq 2$, $\alpha > 1/2$, and $g'(\alpha) = 0$,

$$g(\alpha) = \alpha + \frac{1}{\lambda k} - 1 \geq \alpha - 1/2 > 0.$$

□

We now proceed by characterizing the anonymous equilibria with $\alpha > 1/2$ when $\underline{v}_0 < \underline{v}$ and $\alpha > 1/2$ when $\underline{v}_0 > \underline{v}$. By Lemma A.2 these will be the only such equilibria when $\lambda k \leq 2$. We will consider the other equilibria that can emerge when $\lambda k > 2$ subsequently.

When the neutral bidder's valuation is lower than under strong symmetry, so $\underline{v}_0 < \underline{v}$, the neutral bidder will win less often, which, in turn, raises the stakes for the polarized bidders as the threat of losing to each other increases. This increases their willingness to pay relative to Example 2, making the polarized bidders bid even more aggressively. Consequently, some fraction of neutral bidders will have a lower valuation than any polarized bidder and lose for sure.

In order to characterize the bidding strategies and revenue in Proposition A.1 below, we first establish the following intermediate lemma. This lemma defines a cutoff $\hat{v}_0$, and shows that it is uniquely determined. We will subsequently prove, in Proposition A.1, that $\hat{v}_0$ is the lowest valuation the neutral bidder could have and still win the auction with positive probability.

Lemma A.3. *Suppose $\underline{v}_0 \in (\underline{v} - k/2, \underline{v})$. There exists a unique solution $\hat{v}_0 \in (\underline{v}_0, \infty)$ such that*

$$(\hat{v}_0 - \underline{v}) = \frac{k}{2}\left(1 - e^{-\lambda(\hat{v}_0 - \underline{v}_0)}\right). \tag{9}$$

Moreover $\hat{v}_0 \in (\underline{v}, \underline{v} + k/2)$.

Proof. We first show that there is a solution $\hat{v}_0 \in (\underline{v}_0, \infty)$ to Equation 9 and that this solution is in $(\underline{v}, \underline{v} + k/2)$. Defining

$$h(v) := v - \underline{v} - \frac{k}{2}\left(1 - e^{-\lambda(v - \underline{v}_0)}\right),$$

we have a solution $\hat{v}_0$ to Equation 9 if and only if $h(\hat{v}_0) = 0$. Note that $h(v)$ is continuous and, for any $v \in [\underline{v}_0, \underline{v}]$,

$$h(v) < 0.$$

This follows because, for any $v \in [\underline{v}_0, \underline{v}]$,

$$v - \underline{v} \leq 0 \leq \frac{k}{2}\left(1 - e^{-\lambda(v - \underline{v}_0)}\right),$$

with at least one inequality strict. Similarly, for all $v \geq \underline{v} + k/2$,

$$h(v) \geq k/2 - \frac{k}{2}\left(1 - e^{-\lambda(v - \underline{v}_0)}\right) > 0.$$

It then follows that any solution $\hat{v}_0 \in (\underline{v}_0, \infty)$ must have $\hat{v}_0 \in (\underline{v}, \underline{v} + k/2)$. Moreover, by the intermediate value theorem, there exists a solution in $(\underline{v}, \underline{v} + k/2)$.

We now show that the solution is unique. Differentiating $h(v)$ we get

$$h'(v) = 1 - \frac{\lambda k}{2} e^{-\lambda(v - \underline{v}_0)},$$

and so

$$h''(v) = \frac{\lambda^2 k}{2} e^{-\lambda(v - \underline{v}_0)},$$

which is strictly positive for all $v > \underline{v}_0$. As $h(\cdot)$ is strictly convex, $h(\underline{v}) < 0$ and $h(\underline{v} + k/2) > 0$, it follows that there is a unique $\hat{v}_0 \in (\underline{v}, \underline{v} + k/2)$ such that $h(\hat{v}_0) = 0$.

□

Our next proposition uses the solution to Equation 9 to characterize the equilibrium bidding strategies and revenue.

Proposition A.1. *Suppose $\underline{v}_0 < \underline{v}$, and let $\hat{v}_0 \in (\underline{v}, \underline{v} + k/2)$ be the solution to Equation 9. Then:*

1. *Bidder -1 and 1 each bidding $b_i(\tilde{v}_i) = \frac{k}{2} + \hat{v}_0 + \tilde{v}_i$ and bidder 0 bidding $b_0(\tilde{v}_0) = \frac{k}{2} + \underline{v}_0 + \tilde{v}_0$ constitutes an anonymous equilibrium in weakly undominated strategies.*

2. *In the equilibrium described in part 1, bidder 0 wins with probability $\frac{e^{-\lambda(\hat{v}_0 - \underline{v}_0)}}{4 - e^{-\lambda(\hat{v}_0 - \underline{v}_0)}} < 1/3$ and bidders -1 and 1 each win with probability $\frac{2 - e^{-\lambda(\hat{v}_0 - \underline{v}_0)}}{4 - e^{-\lambda(\hat{v}_0 - \underline{v}_0)}} > 1/3$. The expected revenue is*

$$\frac{k}{2} + \hat{v}_0 + \frac{5}{6\lambda} e^{-\lambda(\hat{v}_0 - \underline{v}_0)} + \frac{1}{2\lambda}\left(1 - e^{-\lambda(\hat{v}_0 - \underline{v}_0)}\right). \tag{10}$$

3. *The strategies described in Part 1 constitute the unique anonymous equilibrium in weakly undominated strategies in which $\alpha \geq 1/2$.*

Proof. We prove each part separately.

Part 1: We show that the specified strategies constitute an anonymous equilibrium in weakly undominated strategies. Consider first the neutral bidder. Since either other bidder winning gives her the same utility she has a weakly dominant strategy to bid $b_0(\tilde{v}_0) = \frac{k}{2} + \underline{v}_0 + \tilde{v}_0$. Therefore bidder 0 is optimizing.

Now consider bidder 1. Note that under the specified strategies, b_{-1} is exponentially distributed on $(k/2 + \hat{v}_0, \infty)$ with rate λ. By the memoryless property, conditional on b_0 exceeding $\hat{v}_0$, b_0 is exponentially distributed on $(k/2 + \hat{v}_0, \infty)$ with rate λ. As b_0 exceeds $k/2 + \hat{v}_0$ with probability $1 - F(\hat{v}_0 - \underline{v}_0)$ it follows that, for any b_1, if bidder 1 doesn't win then, with probability $F(\hat{v}_0 - \underline{v}_0)$, bidder -1 wins for sure, and with probability $1 - F(\hat{v}_0 - \underline{v}_0)$ bidders -1 and 0 each win with equal probability. Therefore

$$Pr(b_{-1} > b_0 | \max\{b_{-1}, b_0\} > b_1) = F(\hat{v}_0 - \underline{v}_0) + (1 - F(\hat{v}_0 - \underline{v}_0))\frac{1}{2} = \frac{2 - e^{-\lambda(\hat{v}_0 - \underline{v}_0)}}{2},$$

which is constant in b_1. Her expected utility of not winning the contract with bid b_1 is then

$$-kPr(b_{-1} > b_0 | \max\{b_{-1}, b_0\} > b_1) = -k\frac{2 - e^{-\lambda(\hat{v}_0 - \underline{v}_0)}}{2}.$$

Note that this is a constant in b_1, for any $b_1 > \hat{v}_0 + k/2$, and so the bid she submits does not affect the expected negative externality she receives if she doesn't win the contract. Hence, by Equation 9, her expected utility of not winning the contract is $-(\hat{v}_0 - \underline{v}) - k/2$ and her best response when her valuation is v_1 is to bid $k/2 + (\hat{v}_0 - \underline{v}) + v_1$. We can then conclude that bidding strategy

$$b_1(\tilde{v}_1) = \frac{k}{2} + \hat{v}_0 + \tilde{v}_1$$

is a best response for bidder 1 and, by symmetry, that $b_{-1}(\tilde{v}_{-1}) = \frac{k}{2} + \hat{v}_0 + \tilde{v}_{-1}$ is a best response for bidder -1. So the specified strategies constitute an anonymous equilibrium in weakly undominated strategies.

Part 2: We now determine the winning probabilities for each bidder and the revenue for the seller in the equilibrium described in part 1. To calculate the winning probabilities note that

$$P(-1 \text{ wins}) = P(-1 \text{ wins}|1 \text{ doesn't})P(1 \text{ doesn't}) = \frac{2 - e^{-\lambda(\hat{v}_0 - \underline{v}_0)}}{2}(1 - P(1 \text{ wins})).$$

Therefore, by symmetry, we have that

$$P(-1 \text{ wins}) = P(1 \text{ wins}) = \frac{2 - e^{-\lambda(\hat{v}_0 - \underline{v}_0)}}{4 - e^{-\lambda(\hat{v}_0 - \underline{v}_0)}}.$$

Finally, we can calculate the probability that 0 wins, since

$$P(0 \text{ wins}) = 1 - 2P(1 \text{ wins}) = \frac{e^{-\lambda(\hat{v}_0 - \underline{v}_0)}}{4 - e^{-\lambda(\hat{v}_0 - \underline{v}_0)}}.$$

We next calculate revenue. Note that the neutral bidder never wins if $v_0 < \hat{v}_0$ but, conditional on having a valuation higher than $\hat{v}_0$, her valuation is exponentially distributed. Hence, revenue is equivalent to a model in which, with probability $F(\hat{v}_0 - \underline{v}_0)$ there are two i.i.d. bidders with valuations $k/2 + \hat{v}_0 + \tilde{v}_i$ where $\tilde{v}_i$ is exponentially distributed with rate λ, and with probability $1 - F(\hat{v}_0 - \underline{v}_0)$ there are three i.i.d. bidders with valuations $k/2 + \hat{v}_0 + \tilde{v}_i$ where $\tilde{v}_i$

is exponentially distributed with rate λ. Consequently, the expected revenue is

$$\frac{k}{2} + \hat{v}_0 + F(\hat{v}_0 - \underline{v}_0)\mathbb{E}[\tilde{v}(2,2)] + (1 - F(\hat{v}_0 - \underline{v}_0))\mathbb{E}[\tilde{v}(2,3)],$$

which, given Equation 3 and Equation 4, is equal to

$$\frac{k}{2} + \hat{v}_0 + \frac{1}{2\lambda}(1 - e^{-\lambda(\hat{v}_0 - \underline{v}_0)}) + \frac{5}{6\lambda}e^{-\lambda(\hat{v}_0 - \underline{v}_0)}.$$

Part 3: We show that the equilibrium described in Part 1 is the unique anonymous equilibrium with $\alpha \geq 1/2$. There is an equilibrium with $\alpha \geq 1/2$ if and only if α solves

$$\begin{aligned}
\alpha &= Pr(b_{-1} > b_0 | \max\{b_{-1}, b_0\} > b_1) \\
&= F((\alpha - 1/2)k + \underline{v} - \underline{v}_0) + \frac{1}{2}(1 - F((\alpha - 1/2)k + \underline{v} - \underline{v}_0)) \\
&= \frac{2 - e^{-\lambda((\alpha - 1/2)k + \underline{v} - \underline{v}_0)}}{2}.
\end{aligned}$$

Defining $v := \underline{v} + (\alpha - 1/2)k$, this we have that v solves

$$v - \underline{v} = k\frac{1 - e^{-\lambda(v - \underline{v}_0)}}{2}.$$

By Lemma A.3 the unique solution with $v > \underline{v}$ to Equation 9 is $\hat{v}_0$. Hence, the bidding strategies described in part 1, in which $\alpha k + \underline{v} = \hat{v}_0$, is the unique anonymous equilibrium in undominated strategies with $\alpha \geq 1/2$.

□

Proposition A.1 consists of three parts. Part 1 characterizes an anonymous equilibrium in which the bids of the polarized bidders first order stochastic dominate those of the neutral bidder. Part 2 shows that in this equilibrium the polarized bidders win more often than the neutral bidder and characterizes the seller's revenue. Part 3 of Proposition A.1 demonstrates that this is the unique equilibrium with $\alpha \geq 1/2$; by Lemma A.2 that makes it the unique anonymous equilibrium when $\lambda k \leq 2$.

We now turn to the case in which the neutral bidder's valuation is higher than under strong symmetry, $\underline{v}_0 > \underline{v} + k/2$, focusing first on the equilibrium with $\alpha < 1/2$. Then, the neutral bidder wins more often, which, in turn, lowers the willingness to pay of the polarized bidders. As a result, some polarized bidders will submit bids that are sure to lose. The following lemma, analogous to Lemma A.3, describes $\hat{v}_1$, which we will demonstrate in Proposition A.2 is the lowest valuation a polarized bidder could have and still win the auction with positive probability.

Lemma A.4. *Suppose $\underline{v}_0 > \underline{v}$. There exists a unique $\hat{v}_1 \in (\underline{v}, \infty)$ such that*

$$\left(\underline{v}_0 + \frac{k}{2} - \hat{v}_1\right) = \frac{k}{2} e^{-\lambda(\hat{v}_1 - \underline{v})}. \tag{11}$$

Moreover, $\hat{v}_1 \in (\underline{v}_0, \underline{v}_0 + k/2)$.

Proof. We first show that there is a solution to Equation 11 and that every solution is in $(\underline{v}_0, \underline{v}_0 + k/2)$. We first define,

$$h(v) := (\underline{v}_0 - v) + \frac{k}{2} - \frac{k}{2} e^{-\lambda(v - \underline{v})},$$

and note that we have a solution to Equation 11 if and only if $h(\hat{v}_1) = 0$. Furthermore, $h(\cdot)$ is continuous in v and for any $v \geq \underline{v}_0 + k/2$,

$$h(v) \leq -\frac{k}{2} e^{-\lambda(v - \underline{v})} < 0,$$

and for any $v \in (\underline{v}, \underline{v}_0]$,

$$h(v) \geq \frac{k}{2} - \frac{k}{2} e^{-\lambda(v - \underline{v})} > 0.$$

Hence, by the intermediate value theorem, there exists $\hat{v}_1 \in (\underline{v}_0, \underline{v}_0 + k/2)$ such that

$$h(\hat{v}_1) = 0.$$

Moreover, every solution to $h(\hat{v}_1) = 0$ must lie in $(\underline{v}_0, \underline{v}_0 + k/2)$.

We now show that there is a unique solution in $(\underline{v}_0, \underline{v}_0 + k/2)$. To see this, note that

$$h'(v) = -1 + \frac{\lambda k}{2} e^{-\lambda(v - \underline{v})},$$

and

$$h''(v) = \frac{-\lambda^2 k}{2} e^{-\lambda(v - \underline{v})} < 0.$$

As $h(\cdot)$ is strictly concave, $h(\underline{v}_0) > 0$ and $h(\underline{v}_0 + k/2) < 0$, there is a unique $\hat{v}_1 \in (\underline{v}_0, \underline{v}_0 + k/2)$ such that $h(\hat{v}_1) = 0$.

□

With $\hat{v}_1$ defined by Equation 11 we can characterize the anonymous equilibria when $\underline{v}_0 > \underline{v}$. Relative to Proposition A.1 there is an additional layer of equilibrium multiplicity because, in a second price auction, a bidder who has a valuation so low they never win would be indifferent over different bids that are sure to lose. While focusing on weakly dominant strategies pins down the bid of the neutral bidder, it does not pin down exactly the bid of the polarized bidders, whose valuation depends on who they expect to win if they don't. While low valuation polarized bidders never win the contract, their bids may influence the winner's payment. We characterize the anonymous equilibria in the following Proposition.

Proposition A.2. *Suppose $\underline{v} < \underline{v}_0$, and let $\hat{v}_1 \in (\underline{v}_0, \underline{v}_0 + k/2)$ solve Equation 11. Then:*

1. *Bidder 0 bidding $b_0(\tilde{v}_0) = k/2 + \underline{v}_0 + \tilde{v}_0$ and bidders -1 and 1 each bidding*

$$b_i(\tilde{v}_i) = \begin{cases} \in \left[\underline{v} + \tilde{v}_i, \frac{k}{2} + \underline{v}_0\right] & \text{if } \tilde{v}_i \leq \hat{v}_1 - \underline{v}, \\ \frac{k}{2} + \underline{v} + \tilde{v}_i + \underline{v}_0 - \hat{v}_1 & \text{if } \tilde{v}_i > \hat{v}_1 - \underline{v}, \end{cases} \tag{12}$$

   constitutes an anonymous equilibrium in weakly undominated strategies.

2. *In an equilibrium of the form described in part 1, bidder 0 wins with probability $\frac{2-e^{-\lambda(\hat{v}_1-\underline{v})}}{2+e^{-\lambda(\hat{v}_1-\underline{v})}} > 1/3$ and bidders -1 and 1 each win with probability $\frac{e^{-\lambda(\hat{v}_1-\underline{v})}}{2+e^{-\lambda(\hat{v}_1-\underline{v})}} < 1/3$. In the revenue maximizing equilibrium in this class, $b_i(\tilde{v}_i) = \underline{v}_0 + k/2$ when $\tilde{v}_i \leq \hat{v}_1 - \underline{v}$ and the seller's revenue is*

$$\frac{k}{2} + \underline{v}_0 - \frac{1}{6\lambda} e^{-2\lambda(\hat{v}_1-\underline{v})} + \frac{1}{\lambda} e^{-\lambda(\hat{v}_1-\underline{v})}. \tag{13}$$

3. *Every anonymous equilibrium in weakly undominated strategies in which $\alpha \leq 1/2$ must take the form described in part 1.*

Proof. We prove each part in turn.

Part 1: We show that the specified strategies constitute an anonymous equilibrium in weakly undominated strategies. First consider bidder 0. Since either other bidder winning gives them the same utility they have a weakly dominant strategy to bid $b_0(\tilde{v}_0) = k/2 + \underline{v}_0 + \tilde{v}_0$ and so bidder 0 is optimizing. Now consider bidder 1, and note that bidder 1 can never win bidding less than $k/2 + \underline{v}_0$. Note also that, under the specified strategies, b_0 is exponentially distributed from $(k/2 + \underline{v}_0, \infty)$. Further, with probability $1 - F(\hat{v}_1 - \underline{v})$, b_{-1} is exponentially distributed from $(k/2 + \underline{v}_0, \infty)$ as well and so both bidder -1 and 0 have the same likelihood of winning; with probability $F(\hat{v}_1 - \underline{v})$, bidder -1 bids low enough to never win regardless of $\tilde{v}_0$. So, if bidder 1 bids b_1 and doesn't win,

$$Pr(b_{-1} > b_0 | \max\{b_{-1}, b_0\} > b_1) = (1 - F(\hat{v}_1 - \underline{v}))\frac{1}{2} + F(\hat{v}_1 - \underline{v})(0) = \frac{e^{-\lambda(\hat{v}_1 - \underline{v})}}{2}.$$

Hence her expected utility if not winning the contract with bid b_1 is

$$-kPr(b_{-1} > b_0 | \max\{b_{-1}, b_0\} > b_1) = -k\frac{e^{-\lambda(\hat{v}_1 - \underline{v})}}{2}.$$

Note that this is a constant in b_1, and so the bid she submits does not affect the expected negative externality she receives from how the contract is allocated. Hence, by Equation 11 her expected utility if not winning the contract is

$$-\frac{k}{2} - \underline{v_0} + \hat{v}_1.$$

Consequently, the net valuation of winning the object for bidder 1 if her valuation is $\underline{v} + \tilde{v}_1$ is

$$\tilde{v_1} + \underline{v} - \hat{v}_1 + \underline{v}_0 + \frac{k}{2}.$$

As the minimum bid that allows her to win is anything over $\underline{v}_0 + k/2$, all bids lower than $\underline{v}_0 + k/2$ are equivalent, though any bid lower than $\underline{v} + \tilde{v}_1$ is weakly dominated. Consequently, bidding

$$b_1(\tilde{v}_1) = \begin{cases} \in \left[\underline{v} + \tilde{v}_1, \frac{k}{2} + \underline{v}_0\right] & \text{if } \tilde{v}_1 \leq \hat{v}_1 - \underline{v}, \\ \frac{k}{2} + \underline{v} + \tilde{v}_1 + \underline{v}_0 - \hat{v}_1 & \text{if } \tilde{v}_1 > \hat{v}_1 - \underline{v}, \end{cases}$$

is a weakly undominated best response for bidder 1. As bidder 1 is optimizing, by symmetry, bidder -1 is too. So we have an anonymous equilibrium in weakly undominated strategies.

Part 2: We first determine the winning probabilities in equilibrium. Note that

$$P(-1 \text{ wins}) = P(-1 \text{ wins} | 1 \text{ doesn't})P(1 \text{ doesn't}) = \frac{e^{-\lambda(\hat{v}_1 - \underline{v})}}{2}(1 - P(1 \text{ wins})),$$

and so by symmetry

$$P(-1 \text{ wins}) = P(1 \text{ wins}) = \frac{e^{-\lambda(\hat{v}_1 - \underline{v})}}{2 + e^{-\lambda(\hat{v}_1 - \underline{v})}},$$

and

$$P(0 \text{ wins}) = \frac{2 - e^{-\lambda(\hat{v}_1 - \underline{v})}}{2 + e^{-\lambda(\hat{v}_1 - \underline{v})}}.$$

We now turn to revenue. As revenue is increasing in the bids of each player, the revenue maximizing equilibrium of the form described in part 1 involves a sure losing polarized bidder bidding $\underline{v}_0+k/2$. Note that bidder 0's bids are exponentially distributed from $(\underline{v}_0+k/2,\infty)$ with rate λ. Similarly, for bidder $i\in\{-1,1\}$ they bid $\underline{v}_0$ with probability $F(\hat{v}_1-\underline{v})$ and with probability $1-F(\hat{v}_1-\underline{v})$ their bids are exponentially distributed from $(\underline{v}_0+k/2,\infty)$ with rate λ.

Recalling that, by Equation 3 and Equation 4, $\mathbb{E}[\tilde{v}(2,3)]=\frac{5}{6\lambda}$ and $\mathbb{E}[\tilde{v}(2,2)]=\frac{1}{2\lambda}$ this implies that the expected revenue is

$$\frac{k}{2}+\underline{v}_0+(1-F(\hat{v}_1-\underline{v}))^2\frac{5}{6\lambda}+2F(\hat{v}_1-\underline{v})(1-F(\hat{v}_1-\underline{v}))\frac{1}{2\lambda},$$

which simplifies to

$$\frac{k}{2}+\underline{v}_0-e^{-2\lambda(\hat{v}_1-\underline{v})}\frac{1}{6\lambda}+e^{-\lambda(\hat{v}_1-\underline{v})}\frac{1}{\lambda}.$$

Part 3: We show that there is a unique $\alpha<1/2$ that is consistent with equilibrium. To see this, note that if $\alpha<1/2$ then

$$\begin{aligned}\alpha&=\frac{1}{2}Pr((\alpha-1/2)k+v_1>\underline{v}_0)\\&=\frac{1-F(\underline{v}_0-\underline{v}-(1/2-\alpha)k)}{2}\\&=\frac{e^{-\lambda(\underline{v}_0-(1/2-\alpha)k-\underline{v})}}{2}.\end{aligned}$$

Defining $v=(1/2-\alpha)k+\underline{v}_0$ this equation reduces to

$$\frac{k}{2}+\underline{v}_0-v=\frac{k}{2}e^{-\lambda(v-\underline{v})},$$

and by Lemma A.4 there is a unique solution to this equation with $v>\underline{v}_0$. We can then conclude there is a unique $\alpha<1/2$ that is consistent with equilibrium.

□

Part 1 characterizes a class of anonymous equilibria in terms of $\hat{v}_1$. In these equilibria, if the private valuation of the polarized bidders is sufficiently low, they submit a bid that's sure to lose. Part 2 then shows that the neutral bidder wins more often than the polarized bidders. As seller revenue is maximized when the polarized bidders submit the highest sure losing bid, the maximal seller revenue in this class of equilibria is characterized in Equation 13.

Finally, we consider other possible anonymous equilibria. Such equilibria exist only when λ and k are large, which is the case in which there is the greatest incentive to exclude neutral bidders. The next lemma shows that when there are multiple equilibria and $\underline{v} < \underline{v}_0$, the revenue is lower than with two polarized bidders.

Lemma A.5. *If $\underline{v} < \underline{v}_0$ then any anonymous equilibrium with three bidders and $\alpha > 1/2$ generates lower revenue than with two polarized bidders.*

Proof. By Lemma A.2 if $\alpha > 1/2$ then it must be that $\lambda k > 2$. Defining $v := \underline{v} + (\alpha - 1/2)k$, we have that v solves

$$v - \underline{v} = k\frac{1 - e^{-\lambda(v - \underline{v}_0)}}{2}.$$

Hence in equilibrium we must have v solve Equation 9 with $v > \underline{v}$. By Lemma A.3 the unique solution is $\hat{v}_0 + k/2$. To calculate the revenue note that the neutral bidder never wins if $v_0 < \hat{v}_0$ but, conditional on having a valuation higher than $\hat{v}_0$, her valuation is exponentially distributed. Hence, revenue is equivalent to a model in which, with probability $F(\hat{v}_0 - \underline{v}_0)$ there are two i.i.d. bidders with valuations $k/2 + \hat{v}_0 + \tilde{v}_i$ and with probability $1 - F(\hat{v}_0 - \underline{v}_0)$ there are three i.i.d. bidders with valuations $k/2 + \hat{v}_0 + \tilde{v}_i$. Consequently, the expected revenue is

$$\frac{k}{2} + \hat{v}_0 + F(\hat{v}_0 - \underline{v}_0)\mathbb{E}[\tilde{v}(2,2)] + (1 - F(\hat{v}_0 - \underline{v}_0))\mathbb{E}[\tilde{v}(2,3)],$$

which, given Equation 3 and Equation 4, is equal to

$$\frac{k}{2} + \hat{v}_0 + \frac{1}{2\lambda}(1 - e^{-\lambda(\hat{v}_0 - \underline{v}_0)}) + \frac{5}{6\lambda}e^{-\lambda(\hat{v}_0 - \underline{v}_0)}.$$

By Proposition 1 the seller's revenue when there are two bidders is

$$k + \mathbb{E}[\tilde{v}(2,2)] = k + \underline{v} + \frac{1}{2\lambda}.$$

Hence the difference between the revenue with two and three bidders in this equilibrium is

$$d(\lambda) := \frac{k}{2} + \underline{v} - \hat{v}_0 - \frac{1}{3\lambda}e^{-\lambda(\hat{v}_0 - \underline{v}_0)},$$

and preventing participation from the neutral bidder raises greater revenue if $d(\lambda) > 0$. Next note that we can re-write

$$d(\lambda) = \frac{k}{2} - k\left(\frac{1}{2} - \frac{e^{-\lambda(\hat{v}_0 - \underline{v}_0)}}{2}\right) - \frac{1}{3\lambda}e^{-\lambda(\hat{v}_0 - \underline{v}_0)}$$
$$= \frac{(3k\lambda - 2)e^{-\lambda(\hat{v}_0 - \underline{v}_0)}}{6\lambda}.$$

As the denominator is always positive, it follows that $d(\lambda) > 0$ if $\lambda > 2/3k$. Given that $\lambda k > 2$ we can conclude that $d(\lambda) > 0$ and so the revenue is less than in the equilibrium with $B = \{-1, 1\}$. □

With these results in hand we now prove Proposition 4 and Proposition 5.

Proof of Proposition 4. Immediate from Lemma A.1. □

Proof of Proposition 5. We first note that, by Proposition 1, the seller's revenue when there are two bidders is

$$k + \mathbb{E}[\tilde{v}(2,2)] = k + \underline{v} + \frac{1}{2\lambda}.$$

Consider first the case in which $\underline{v}_0 < \underline{v}$. Using the characterization in Proposition A.1, the difference between the revenue with two and three bidders in the unique equilibrium with $\alpha \geq 1/2$ is

$$d(\lambda) := \frac{k}{2} + \underline{v} - \hat{v}_0 - \frac{1}{3\lambda}e^{-\lambda(\hat{v}_0 - \underline{v}_0)},$$

and preventing participation from the neutral bidder raises greater revenue if $d(\lambda) > 0$ and less revenue if $d(\lambda) < 0$. Next note that by Equation 9 we can re-write

$$d(\lambda) = \frac{k}{2} - k\left(\frac{1}{2} - \frac{e^{-\lambda(\hat{v}_0 - \underline{v}_0)}}{2}\right) - \frac{1}{3\lambda}e^{-\lambda(\hat{v}_0 - \underline{v}_0)}$$
$$= \frac{(3k\lambda - 2)e^{-\lambda(\hat{v}_0 - \underline{v}_0)}}{6\lambda}.$$

It follows that $d(\lambda) > 0$ if $\lambda > 2/3k$ and $d(\lambda) < 0$ if $\lambda < 2/3k$.

Finally, note that if $\alpha < 1/2$ the bid of each polarized bidder is lower than when $\alpha > 1/2$, and so, if an equilibrium with $\alpha < 1/2$ exists it generates lower revenue than the equilibrium considered. Hence, when $\lambda > 2/3k$, the revenue is higher with two polarized bidders than in any anonymous equilibrium with three bidders; when $\lambda < 2/3k$ the revenue is higher in the unique anonymous equilibrium with three bidders.

Now consider the case in which $\underline{v}_0 \in (\underline{v}, \underline{v} + k/2)$. We first show that there exists a $\overline{\lambda} > 0$ such that, the revenue in part 2 of Proposition A.2, is higher (lower) than with two polarized bidders if and only if $\lambda < \overline{\lambda}$ ($\lambda > \overline{\lambda}$). By Equation 13, the revenue is higher from restricting participation if and only if

$$d(\lambda) := \frac{k}{2} + \underline{v} - \underline{v}_0 + \frac{1}{2\lambda} + e^{-2\lambda(\hat{v}_1 - \underline{v})}\frac{1}{6\lambda} - e^{-\lambda(\hat{v}_1 - \underline{v})}\frac{1}{\lambda}$$

is positive.

Note first that

$$\lim_{\lambda \to \infty} d(\lambda) = \frac{k}{2} + \underline{v} - \underline{v}_0 > 0,$$

and so there exists a $\overline{\lambda}$ such that it is better to prevent participation by the neutral bidder if $\lambda > \overline{\lambda}$.

Similarly, since

$$d(\lambda) = \frac{k}{2} + \underline{v} - \underline{v}_0 + \frac{1}{6\lambda}(3 + e^{-2\lambda(\hat{v}_1 - \underline{v})} - 6e^{-\lambda(\hat{v}_1 - \underline{v})}),$$

and

$$\lim_{\lambda \to 0}(3 + e^{-2\lambda(\hat{v}_1 - \underline{v})} - 6e^{-\lambda(\hat{v}_1 - \underline{v})}) = -2,$$

it follows that

$$\lim_{\lambda \to 0} d(\lambda) = -\infty,$$

and so it is unprofitable to exclude the neutral bidder when λ is sufficiently small.

To show that there exists a unique threshold $\overline{\lambda}$ such that is it revenue increasing (decreasing) to exclude the neutral bidder if $\lambda > \overline{\lambda}$ ($\lambda < \overline{\lambda}$) it remains to show that $d(\lambda) = 0$ has a unique solution. To prove this it is sufficient to show that $d'(\lambda) > 0$ for any λ such that $d(\lambda) = 0$.

We first note that, implicitly differentiating Equation 11, it follows that

$$\frac{\partial \hat{v}_1}{\partial \lambda} = \frac{\frac{k}{2}e^{-\lambda(\hat{v}_1 - \underline{v})}(\hat{v}_1 - \underline{v})}{1 - \lambda\frac{k}{2}e^{-\lambda(\hat{v}_1 - \underline{v})}(\hat{v}_1 - \underline{v})} > -\frac{\hat{v}_1 - \underline{v}}{\lambda},$$

and so

$$\frac{\partial[\lambda(\hat{v}_1 - \underline{v})]}{\partial \lambda} = \hat{v}_1 - \underline{v} + \lambda\frac{\partial \hat{v}_1}{\partial \lambda} > 0.$$

Differentiating $d(\lambda)$ we see that

$$d'(\lambda) = -\frac{1}{6\lambda^2}(3 + e^{-2\lambda(\hat{v}_1 - \underline{v})} - 6e^{-\lambda(\hat{v}_1 - \underline{v})}) + \frac{1}{6\lambda}(-2e^{-2\lambda(\hat{v}_1 - \underline{v})} + 6e^{-\lambda(\hat{v}_1 - \underline{v})})\frac{\partial[\lambda(\hat{v}_1 - \underline{v})]}{\partial \lambda}.$$

Note that the second term is positive, and since $d(\lambda) = 0$ implies that

$$3 + e^{-2\lambda(\hat{v}_1 - \underline{v})} - 6e^{-\lambda(\hat{v}_1 - \underline{v})} < 0,$$

the first term is positive when when $d(\lambda) = 0$. We can then conclude that $d'(\lambda) > 0$ when $d(\lambda) = 0$.

Finally, consider equilibria of the form not in Proposition A.2. Such an equilibrium must involve $\alpha > 1/2$, and so by Lemma A.5 must generate lower revenue than two polarized bidders. As any equilibrium with $\alpha > 1/2$ generates higher revenue than any equilibrium with $\alpha \leq 1/2$, it follows that no equilibrium with $\alpha > 1/2$ can exist when $\lambda < \overline{\lambda}$. Hence we can conclude that when $\lambda > \overline{\lambda}$ the seller's revenue is higher with $B = \{-1, 0, 1\}$ and if $\lambda < \overline{\lambda}$ the revenue maximizing anonymous equilibrium when $B = \{-1, 0, 1\}$ gives higher revenue than the equilibrium with $B = \{-1, 1\}$. □

Proof of Proposition 6. Recall that no anonymous equilibrium with $\alpha \geq 1/2$ can exist when $\lambda k \leq 2$ and there is a unique $\alpha < 1/2$ consistent with equilibrium for any λ. When $\lambda k > 2$ there can exist other equilibria, but they must involve $\alpha > 1/2$. Hence taking a selection of equilibria that are continuous in λ must have $\alpha < 1/2$ for all λ. Hence this selection criteria chooses an anonymous equilibrium of the form characterized in Proposition A.2.

By Proposition A.2, the neutral bidder wins with probability $\frac{2-e^{-\lambda(\hat{v}_1 - \underline{v})}}{2+e^{-\lambda(\hat{v}_1 - \underline{v})}}$. Moreover, by Equation 11,

$$\lim_{\lambda \to \infty} \hat{v}_1 = \underline{v}_0 + \frac{k}{2} > \underline{v}.$$

Hence,

$$\lim_{\lambda \to \infty} P(0 \text{ wins}) = 1.$$

Consequently, for any $\varepsilon > 0$, there exists a $\hat{\lambda}(\varepsilon)$ such that, for all $\lambda > \hat{\lambda}(\varepsilon)$,

$$P(0 \text{ wins}) > 1 - \varepsilon.$$

Since, by Proposition A.2 and Remark 1, whenever bidder 0 wins it is efficient, it follows that the equilibrium is efficient with probability greater than $1 - \varepsilon$. However, defining $\lambda^*(\varepsilon) = \max\{\hat{\lambda}(\varepsilon), \overline{\lambda}\}$, it follows from Proposition 5 that the seller will prevent the neutral bidder from participating when $\lambda > \lambda^*(\varepsilon)$. $\square$

Proof of Proposition 7. This is equivalent to the baseline model when the polarized bidders have valuations drawn exponentially from $(\underline{v} - c, \infty)$. Hence the result follows from Proposition 5. $\square$

A.3. Proofs of Section 5

Proof of Lemma 1. Fix an auction that satisfies $\mathcal{FS}$ and let $\varepsilon > 0$. Assume that $F(\cdot)$ is twice continuously differentiable and define $\delta = \mathbb{E}[\tilde{v}_i]$.

Define $W_i(\hat{B})$ to be expected utility of bidder i when $\hat{B} \subseteq \{-1, 0, 1\}$ is the set of bidders who attend the auction given that bidders $B = \{-1, 0, 1\}$ are invited. Define $R(\hat{B})$ as the seller's expected revenue given $\hat{B}$. Note that the expected revenue of the seller from $B = \{-1, 0, 1\}$ is then

$$\mathbb{E}[R] = \sum_{B' \subseteq B} Pr[\hat{B} = B']R(B').$$

First note that by Remark 3 the neutral bidder will attend, and so any B' with $Pr[\hat{B} = B'] > 0$ must have $0 \in B'$. It is then sufficient to consider the incentives to enter the auction for the polarized bidders. If the neutral bidder is the only bidder to attend ($\hat{B} = \{0\}$), she wins the contract at price 0 and the payoff to each polarized bidder is 0. That is that $W_0(\{0\}) = \mathbb{E}[v_0] = \underline{v}_0 + \delta$ and $W_{-1}(\{0\}) = W_1(\{0\}) = R(\{0\}) = 0$.

We begin by characterizing the payoffs and revenue in the second and first price auction when the neutral bidder and one polarized bidder attend the auction. WLOG take the polarized bidder to be bidder 1 so $\hat{B} = \{0, 1\}$. In this setting the net valuation in the auction of the neutral bidder is $v_0 + k/2$ and the net valuation of bidder 1 is v_1. We now define

$$\varepsilon_1 = \min\left\{\frac{\varepsilon}{6}, \underline{v}_0 + \frac{k}{2} - \underline{v}\right\} > 0, \tag{14}$$

and show that in either the first or second price auction there exists a $\delta_1 > 0$ such that, for all $\delta < \delta_1$,

$$W_i(\{0, 1\}) \in (-\varepsilon_1, \varepsilon_1), \tag{15}$$

for $i \in \{-1, 1\}$ and

$$R(\{0, 1\}) < \underline{v} + \varepsilon_1. \tag{16}$$

Consider first the second price auction. In the second price auction, bidder 0 has a weakly dominant strategy to bid v_0, bidder 1 has a weakly dominant strategy to bid v_1, and the higher bid winning and the price equal to the second highest bid. The seller's expected revenue is then

$$R(\{0,1\}) = \mathbb{E}[\min\{v_0 + k/2, v_1\}] \leq \mathbb{E}[v_1] = \underline{v} + \mathbb{E}[\tilde{v}_i] = \underline{v} + \delta.$$

Note also that, since the bid of bidder 0 is always at least $\underline{v}_0 + k/2 > \underline{v}$, the payoff of bidder 1 is

$$W_1(\{0,1\}) < \delta.$$

Finally, as a necessary condition for bidder 1 to win is that $\tilde{v}_1 > \underline{v}_0 + k/2 - \underline{v}$ and $\delta = \mathbb{E}[\tilde{v}_1] > (1 - F(\underline{v}_0 + k/2 - \underline{v}))(\underline{v}_0 + k/2 - \underline{v})$, it follows that bidder 1's probability of winning is less than

$$\frac{\delta}{\underline{v}_0 + k/2 - \underline{v}}.$$

Hence the payoff of bidder -1 is

$$W_{-1}(\{0,1\}) > -\frac{\delta}{\underline{v}_0 + k/2 - \underline{v}}k.$$

We now define

$$\delta_1^S = \min\left\{\varepsilon_1, \frac{\varepsilon_1(2\underline{v}_0 + k - 2\underline{v})}{2k}\right\} > 0. \tag{17}$$

It then follows that, if the auction is equivalent to the second price auction with two bidders and $\delta < \delta_1^S$, then (15) and (16) both hold.

Now consider the first price auction with $\hat{B} = \{0,1\}$. First note that it follows from by Propositions 1 and 5 of Maskin and Riley (2000) that an equilibrium exists, and every equilibrium must involve monotone bidding strategies. Further, by Lemma 3 of Maskin and Riley (2003), neither bidder will ever bid less than $\underline{v}$. This immediately implies that in equilibrium

$$W_1(0,1) \leq \delta,$$

and

$$W_0(0,1) + k/2 \leq \underline{v}_0 + k/2 + \delta - \underline{v}.$$

We next construct a lower bound on the probability that bidder 0 wins the contract. Denote by $w_0(\tilde{v}_0, b_0)$ the payoff to bidder 0 from bidding b_0 given realization $\tilde{v}_0$. It then follows that

$$w_0(\tilde{v}_0, b_0) + k/2 \geq (\underline{v}_0 + k/2 + \tilde{v}_0 - b_0)Pr(b_1 < b_0) \geq (\underline{v}_0 + k/2 + \tilde{v}_0 - b_0)F(b_0 - \underline{v}),$$

where the second inequality follows because bidder 1 will never bid higher than $v_1 = \underline{v} + \tilde{v}_1$. Evaluating at $\tilde{v}_0 = 0$ we have that

$$w_0(0, b_0) + k/2 \geq (\underline{v}_0 + k/2 - b_0)F(b_0 - \underline{v}).$$

Now let $b_0 > \underline{v}$ be arbitrary, and note that

$$(1 - F(b_0 - \underline{v}))(b_0 - \underline{v}) < \mathbb{E}[\tilde{v}_i] = \delta$$

and so

$$w_0(0, b_0) + k/2 \geq (\underline{v}_0 + k/2 - b_0)\left(1 - \frac{\delta}{b_0 - \underline{v}}\right).$$

Notice that this lower bound on bidder 0's payoff holds across all distributions for which $\delta = \mathbb{E}[\tilde{v}_i]$. Notice also that this lower bound depends continuously on δ. Focusing on the case in which δ is small we have

$$\lim_{\delta \to 0} w_0(0, b_0) + k/2 \geq \lim_{\delta \to 0}(\underline{v}_0 + k/2 - b_0)\left(1 - \frac{\delta}{b_0 - \underline{v}}\right) = \underline{v}_0 + k/2 - b_0,$$

uniformly for all $F(\cdot)$ with $\delta = E[\tilde{v}]$. As $b_0(0) = \arg\max_{b_0 \in [\underline{v}, \underline{v}_0]} w_0(0, b_0)$, we can then conclude that

$$\lim_{\delta \to 0} w_0(0, b_0(0)) = \underline{v}_0 - \underline{v},$$

which implies that

$$\lim_{\delta \to 0} Pr(b_1 < b_0(0)) = 1,$$

and

$$\lim_{\delta \to 0} b_0(0) = \underline{v}.$$

Furthermore, since the probability of winning with $b_0(0)$ approaches 1, and $b_0(\cdot)$ is monotonic, the probability of bidder 0 winning the contract approaches one for any realization $\tilde{v}_0$ as $\delta \to 0$. Finally, as $\tilde{v}_i$ is bounded and bidder 0 can win with probability close to 1 by bidding $b_0(0)$,

$$\lim_{\delta \to 0} \mathbb{E}[b_0(\tilde{v}_0)] = \underline{v}.$$

This, in turn, implies that $\lim_{\delta \to 0} R(\{0, 1\}) = \underline{v}$.

As the utility of bidders $i \in \{-1, 1\}$ are 0 when the neutral bidder wins the contract, we can conclude that there exists a $\delta_1^F > 0$ such that, if the auction is strategically equivalent to a first price auction, then for any $F(\cdot)$ with $\delta = \mathbb{E}[\tilde{v}_1] < \delta_1^F$, (15) and (16) both hold.

Now define

$$\overline{\delta} = \min\{\delta_1^S, \delta_1^F\}. \tag{18}$$

We then have that for all $\delta \in (0, \overline{\delta})$, for any auction mechanism that satisfies $\mathcal{FS}$, the payoff of each bidder $i \in \{-1, 1\}$ is $W_i(\{0,1\}) \in (-\varepsilon_1, \varepsilon_1)$ and $R(\{0,1\}) < \underline{v} + \varepsilon_1$. Note that by a change of variables it also follows that $W_i(\{-1,0\}) \in (-\varepsilon_1, \varepsilon_1)$ for $i \in \{-1,1\}$ and $R(\{-1,0\}) < \underline{v} + \varepsilon_1$ when $\delta < \overline{\delta}$.

We now consider the decision of a polarized bidder to enter the auction under the first and second price auction when $\delta < \overline{\delta}$. Let α_{-1} and α_1 be the probability that bidders -1 and 1 attend the auction when $B = \{-1, 0, 1\}$ are invited. Then note that if $\min\{\alpha_{-1}, \alpha_1\} = 0$ then $\mathbb{E}[R] \le R(\{0,1\}) < \underline{v} + \varepsilon_1 < \underline{v}_0 + k/2$.

Now consider a possible equilibrium in which $\min\{\alpha_{-1}, \alpha_1\} > 0$. Since bidder 1 is choosing $\alpha_1 > 0$ we must have that the expected payoff from attending the auction,

$$\alpha_{-1} W_1(\{-1,0,1\}) + (1 - \alpha_{-1}) W_1(\{0,1\}),$$

is at least as high as from not attending

$$\alpha_{-1} W_1(\{-1,0\}) + (1 - \alpha_{-1}) W_1(\{0\}).$$

Given (15) this implies that

$$\begin{aligned} \alpha_{-1} W_1(\{-1,0,1\}) &\ge \alpha_{-1} W_1(\{-1,0\}) + (1-\alpha_{-1}) W_1(\{0\}) - (1-\alpha_{-1}) W_1(\{0,1\}) \\ &= \alpha_{-1} W_1(\{-1,0\}) - (1-\alpha_{-1}) W_1(\{0,1\}) \\ &\ge -\varepsilon_1. \end{aligned} \tag{19}$$

Similarly,

$$\alpha_1 W_{-1}(\{-1,0,1\}) \ge -\varepsilon_1.$$

Finally, it is immediate that $W_0(\{-1,0,1\}) \ge -k/2$. We will use these lower bounds on the payoffs of the bidders when all three are included to derive an upper bound on the seller's revenue.

For any set of bidders, $\hat{B}$, let γ_i denote the probability of winning the contract for bidder i, and t_i be bidder i's expected payment. It follows that

$$\begin{aligned} W_1(\{-1,0,1\}) &= \gamma_1(\underline{v} + \mathbb{E}[\tilde{v}_1 | 1 \text{ wins}]) - t_1 - k\gamma_{-1} \\ &\leq \gamma_1 \underline{v} + \delta - t_1 - k\gamma_{-1}. \end{aligned}$$

Similarly

$$W_{-1}(\{-1,0,1\}) \leq \gamma_1 \underline{v} + \delta - t_{-1} - k\gamma_{-1},$$

and

$$W_0(\{-1,0,1\}) \leq \gamma_0 \underline{v}_0 + \delta - t_0.$$

Given (19) it then follows that

$$t_1 \leq \gamma_1 \underline{v} + \delta + \frac{\varepsilon_1}{\alpha_{-1}},$$

and similarly we have that

$$t_{-1} \leq \gamma_{-1} \underline{v} + \delta + \frac{\varepsilon_1}{\alpha_1}.$$

Finally, it is immediate that

$$t_0 \leq \gamma_1 \underline{v}_0 + \delta + k/2.$$

We can now calculate the seller's expected revenue from $\hat{B} = \{-1,0,1\}$ when $\delta < \overline{\delta}$. Note that by Equation 14, Equation 17, and Equation 18 we have $\overline{\delta}$ and ε_1 are less than $\varepsilon/6$ and so

$$\begin{aligned} R(\{-1,0,1\}) &= t_{-1} + t_0 + t_1 && (20) \\ &\leq (\gamma_{-1} + \gamma_0 + \gamma_1)\underline{v}_0 + k/2 + 3\delta + \frac{\varepsilon_1}{\alpha_1} + \frac{\varepsilon_1}{\alpha_{-1}} \\ &\leq \underline{v}_0 + k/2 + 3\delta + 2\frac{\varepsilon_1}{\alpha_1\alpha_{-1}} \\ &< \underline{v}_0 + k/2 + \frac{\varepsilon}{2} + \frac{\varepsilon}{3\alpha_1\alpha_{-1}} \\ &\leq \underline{v}_0 + k/2 + \frac{5\varepsilon}{6\alpha_1\alpha_{-1}}. \end{aligned}$$

Using (16) and (20) we can then conclude that for all $\delta \in (0, \overline{\delta})$,

$$\begin{aligned} \mathbb{E}[R] &\leq (1 - \alpha_1\alpha_{-1})R(\{0,1\}) + \alpha_{-1}\alpha_1 R(\{-1,0,1\}) \\ &\leq \underline{v}_0 + k/2 + \frac{\varepsilon}{6} + \alpha_{-1}\alpha_1[R(\{-1,0,1\}) - \underline{v}_0] \\ &< \underline{v}_0 + k/2 + \varepsilon \end{aligned}$$

as claimed. □

Proof of Proposition 8. Combining Remark 3, Proposition 1, and Proposition 3, the seller's expected revenue from $B = \{-1, 1\}$ is $k + \underline{v} + \mathbb{E}[v(2,2)] > k + \underline{v}$. Finally, defining $\varepsilon = \underline{v} + k/2 - \underline{v}_0 > 0$, the result follows from Lemma 1. □

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